Lecture 9

23 Sep 2010

Determinants (Ch. 3)

Recall: The 1×1 matrix $[a]$ is invertible $\iff$ $a \neq 0$

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff$ $ad-bc \neq 0$

We say $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

"determinant"

How about bigger matrices?

First, 3×3.

Given a 3×3 matrix A:

Let C_{ij} (i,j from 1 to 3) be $(-1)^{i+j}$ times the determinant of the 2×2 matrix you get by deleting from A the i^{th} row and the j^{th} column.

Ex If $A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 3 & -4 \\ 1 & 5 & 6 \end{bmatrix}$ then $C_{11} = (-1)^2 \begin{vmatrix} 3 & -4 \\ 5 & 6 \end{vmatrix} = 38$

$$C_{12} = (-1)^3 \begin{vmatrix} 0 & -4 \\ 1 & 6 \end{vmatrix} = -(+4) = -4$$

(Trick for keeping track of $(-1)^{i+j}$: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$)

Then we can define the determinant of A, $\det A$, by the formula:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>Ex</u> Say $A = \begin{bmatrix} 1 & 8 & 4 \\ 0 & 2 & 2 \\ -1 & -2 & 1 \end{bmatrix}$

$$C_{11} = + \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6 \qquad C_{12} = - \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = -2 \qquad C_{13} = + \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} = 2$$

So $\underline{\det A} = 1\cdot 6 + 8\cdot(-2) + 4\cdot 2 = \underline{-2}$

For an $n\times n$ matrix A, $\det A$ is defined in essentially the same way:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

(here C_{ij} are $(-)^{i+j}$ times determinants of $(n-1)\times(n-1)$ matrices — delete i^{th} row and j^{th} col from A)

<u>Ex</u> $A = \begin{bmatrix} 3 & 1 & 0 & 4 \\ -2 & 6 & 1 & 1 \\ 4 & 5 & -1 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

$$\det A = +3\begin{vmatrix} 6 & 1 & 1 \\ 5 & -1 & 0 \\ 3 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 & 1 \\ 4 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} -2 & 6 & 1 \\ 4 & 5 & 0 \\ 0 & 3 & 0 \end{vmatrix} - 4\begin{vmatrix} -2 & 6 & 1 \\ 4 & 5 & -1 \\ 0 & 3 & 0 \end{vmatrix}$$

$$= \cdots$$

$$= 3\cdot 3 - 1\cdot 0 + 0 - 4\cdot 6$$

$$= \underline{-15}$$

Fact:

More generally, to calculate $\det A$, you can use any row or column to do the cofactor expansion (so far I always used the top row)

So, $\det A = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}$

or $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

for any j.

Ex $A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Use 2^{nd} row: $\det A = -3\begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix}$

$= -3 \cdot 16 + 0 - 1 \cdot (-6)$

$= -48 + 6 = \underline{-42}$

Use 2^{nd} col: $\det A = -4\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

$= -4(13) - 2(-5) = -52 + 10 = \underline{-42}$

In ptc, note that if A has a row/col of all 0's then $\det A = 0$.

Ex

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det A = 4 \cdot \begin{vmatrix} 7 & 0 \\ 0 & 3 \end{vmatrix} + 0 + 0$$

$$= 4 \cdot 7 \cdot 3$$ (the product of the diagonal entries)

Ex

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det A = 5 \cdot \begin{vmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{vmatrix} + 0 + 0 + 0$$

$$= 5 \cdot 4 \cdot 7 \cdot 3$$ (product of diag. entries)

Fact: If A is diagonal, $\det A$ is the product of the diagonal entries of A.

$$\begin{vmatrix} a_1 & 0 & 0 \cdots 0 \\ 0 & \ddots & 0 \cdots 0 \\ \vdots & & \ddots & 0 \\ 0 \cdots 0 & & & a_n \end{vmatrix} = a_1 a_2 \cdots a_n$$

Behavior of Determinants under Row Reduction:

Fact: • If A and B are related by adding a multiple of one row to another, then

$$\det A = \det B$$

Ex
$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix}$$

- If A and B are related by swapping two rows then $\det A = -\det B$

Ex
$$\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix}$$

- If one row of A is multiplied by k to get B, then $\det B = k \cdot \det A$

Ex
$$\begin{vmatrix} 1 & 4 & 2 \\ 0 & 3 & 3 \\ 5 & 6 & 2 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 5 & 6 & 2 \end{vmatrix}$$

Q: What if we multiply the whole matrix A by a constant?

Is $\det(kA) = k \det(A)$? No:

Ex $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

$$\det kA = k^2 ad - k^2 bc$$
$$= k^2(ad - bc)$$
$$= k^2 \det A$$

For an $n \times n$ matrix A, $\quad \det(kA) = k^n \det A$

Ex

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2 \cdot \left(1 \cdot 3 \cdot (-6) \cdot 1\right) = \underline{-36}$$

For any invertible A, by the same kind of computation (row reduction) we'll get $\det A = (\text{nonzero constant}) \cdot \det(\text{diagonal matrix})$

And the diagonal matrix that appears here will have all diagonal entries nonzero (because they are the pivots). So the determinant of this diagonal matrix is nonzero!

So, <u>Fact</u>: If A is invertible then $\det A \neq 0$.

Similarly, <u>Fact</u>: If A is <u>not</u> invertible, $\det A = 0$.

Suppose A, B both $n \times n$ matrices.

<u>Fact</u>: $\det(AB) = \det(A)\det(B)$.

(So even though $AB \neq BA$ usually, $\det(AB) = \det(BA)$.)

Ex

$$\det\left(\begin{bmatrix}1&4&3\\0&2&6\\0&0&2\end{bmatrix}\cdot\begin{bmatrix}6&0&0\\1&2&0\\2&8&1\end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix}1&4&3\\0&2&6\\0&0&2\end{bmatrix}\right)\cdot\det\left(\begin{bmatrix}6&0&0\\1&2&0\\2&8&1\end{bmatrix}\right)$$

$$\downarrow RR \qquad\qquad \downarrow RR$$

$$=\det\left(\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}\right)\cdot\det\left(\begin{bmatrix}6&0&0\\0&2&0\\0&0&1\end{bmatrix}\right)$$

$$= 4\cdot 12$$

$$= 48$$