APPENDIX B

Appendix to Chapter 3

The study of mathematics goes well beyond a knowledge of the real numbers. In this chapter, we discuss some topics which are tangentially related to the material of this course, but which can take a willing student a bit further into more advanced mathematics.

1. Cardinality

Intuitively, cardinality is the major of the 'size' of a set. For example, we should predict that a set containing two elements (regardless of what they are) should be 'larger' in some essential way than a set containing only one. In fact, mathematicians were surprised to discover that an infinite set can actually be essentially larger than another, a fact we will prove.

Definition. If A and B are sets, we write $|A| = |B|$ if there exists a bijection $f : A \to B$. We read $|A| = |B|$ as "the cardinality of A equals the cardinality of B.
Note. By Theorem 2.7 in Chapter 2, $|A| = |B| \Rightarrow |B| = |A|$.

Again $|A| = |B|$ intuitively means that both sets have the "same number of elements". This is not startling for finite sets. It is no surprise that $|\{a, b, c\}| = |\{1, 2, 3\}|$. However this definition can lead to non-intuitive results. We can have $A \subseteq B$, $A \neq B$ yet $|A| = |B|$ (how?).

Definition. A is *finite* if $A = \emptyset$ or if there exists $n \in \mathbb{N}$ with $|A| = |\{1, 2, \ldots, n\}|$. (We then say $|A| = 0$ or $|A| = n$ accordingly.) A is *infinite* if A is not finite. A is *countably infinite* if $|A| = |\mathbb{N}|$. A is *countable* if A is finite or countably infinite.

Are all infinite sets also countably infinite?

B.1. Prove that a set A is

(1) countably infinite if and only if we can write $A = \{a_1, a_2, \ldots\}$ where $a_i \neq a_j$ if $i \neq j$.
(2) countably infinite if and only if A is infinite and we can write $A = \{a_1, a_2, \ldots\}$.
(3) countable if and only if $A = \emptyset$ or we can write $A = \{a_1, a_2, \ldots\}$.

Deduce that if $B \subseteq A$ and A is countable, then B is countable.

B.2. Let $|A| = |B|$ and $|B| = |C|$. Prove that $|A| = |C|$.

B.3. Prove that
(1) $|\mathbb{N}| = |\{2, 4, 6, 8, \ldots\}|$
(2) $|\mathbb{N}| = |\mathbb{Z}|$
(3) $|\mathbb{N}| = |\{x \in \mathbb{Q} : x > 0\}|$
(4) $|\mathbb{N}| = |\mathbb{Q}|$.

Hint: For the third part, assuming $a_n > 0$ turn

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2}$$

into an equivalent condition on a quadratic polynomial. Proceed by induction.

B.4. If A is countable and B is countable prove that $A \times B$ is countable.

Hint: You want to construct a list of all elements in $A \times B$ (see 1.17). Can you make an infinite matrix of these elements starting with

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & \ldots \\ \hline b_1 & & & & \\ b_2 & & & & \\ b_3 & & & & \\ \vdots & & & & \end{array}$$

Can you take this matrix and make a list as in 1.17c)?

Our next problem is due to G. Cantor. It is a famous result which shook the mathematical world and has found its way into numerous "popular" math/science books. Cantor went insane. The problem's solution relies on the decimal representation of a real number. In turn this actually involves the notion of convergence of a sequence of reals which we address in chapter 3. But you can use it here. ($1/3 = .333...$ means that $1/3 = \lim_{n\to\infty} x_n$ where $x_n = .33...3$ (n entries). Beware of this fact: Some numbers have 2 decimal representations, e.g., 1 = 1.000... = .999... . This can only happen to numbers which can be represented as decimals with 9 repeating forever from some point.)

B.5. In this problem, we consider the cardinality of $\mathbb{R}$.

(1) Prove that $(0,1)$ is not countable.
(2) Show that $|(0,1]| = |[0,1]|$.
(3) If $a < b$ show that $|(0,1)| = |(a,b)| = |[0,1]| = |[a,b]|$.

Hint: For the first part, assume that $(0,1)$ is countable. Then we can list $(0,1) = \{a_1, a_2, a_3, \ldots\}$. Write each a_i as a decimal to get an infinite matrix as the following example illustrates.

$$\begin{aligned} a_1 &= 0.13974\cdots \\ a_2 &= 0.000002\cdots \\ a_3 &= 0.55556\cdots \\ a_4 &= 0.345587\cdots \\ a_5 &= 0.9871236\cdots \\ &\vdots \end{aligned}$$

Can you find a decimal in $(0,1)$ that is not on this list? Can you describe an algorithm for producing such a number?

We prove in the main body of the text that irrationals exist, but here we can prove much more.

B.6. Prove the following.

(1) If A and B are countable then $A \cup B$ is countable.
(2) $\mathbb{R} \setminus \mathbb{Q}$ is *uncountable* (i.e., not countable).
(3) If $a < b$ then $(a,b) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$.
(4) Prove that if I is countable and for all $i \in I$, A_i is a countable set then $\bigcup_{i \in I} A_i$ is countable.

So irrationals do exist. Does this proof give you any explicit number in $\mathbb{R} \setminus \mathbb{Q}$?

We have not defined $|A| \leq |B|$ yet.

B.7. Give a definition for $|A| \leq |B|$. Your definition should satisfy

(1) $|A| \leq |A|$
(2) $|A| \leq |B|$ and $|B| \leq |C|$ implies that $|A| \leq |C|$.

B.8. Suppose A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$.

Hint: This seemingly trivial statement is actually quite challenging to prove.

Definition. $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Definition. If A is a set, the **power set of** A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Thus, for example,

$$\mathcal{P}(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{3\}\}.$$

B.9. Prove that for all sets A, $|A| < |\mathcal{P}(A)|$.

Hint: Show there does not exist a function $f : A \to \mathcal{P}(A)$ which is onto by assuming such an f exists and considering $B \in \mathcal{P}(A)$ where $B = \{a \in A : a \notin f(a)\}$.

The previous problem demonstrates that there is not largest cardinal.

2. Open and Closed Sets

We have remarked already that the material of the previous section belongs to the branch of mathematics known as set theory. The material of this section, belongs to a branch known as topology. Roughly speaking, topology might be considered the study of what it means for two elements of a set to be "close to each other." We will restrict ourselves exclusively to the topological properties of $\mathbb{R}$, but topology is a very rich subject whose objects goes well beyond merely the real numbers.

Definition. Let $\epsilon > 0$. The interval $(a - \epsilon, a + \epsilon)$ is said to be an *open interval centered at a of radius ϵ.*

B.10. Let $a < b$. Show that (a, b) is an open interval of radius $\epsilon$for some $\epsilon > 0$. What is the center? What is ϵ?

Definition. Let $S \subseteq \mathbb{R}$.

(1) S is *open* if for all $a \in S$ there exists $\epsilon > 0$ with $(a-\epsilon, a+\epsilon) \subseteq S$
(2) S is *closed* if $C(S) = \mathbb{R} \setminus S$ is open.

Is every $S \subseteq \mathbb{R}$ either open or closed? Can you justify your answer?

B.11. Prove that every open interval is an open set and every closed interval is a closed set.

B.12. Classify as open, closed, both or neither

(1) $\emptyset$
(2) $[0,1]$
(3) $\mathbb{Q}$
(4) $\mathbb{R} \setminus \mathbb{Q}$
(5) $\mathbb{R}$
(6) $[0,1] \cup [2,3]$
(7) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Definition. Let $S \subset \mathbb{R}$.

(1) $x \in \text{int}(S)$ if there exists $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \subseteq S$.
(2) $x \in \text{bd}(S)$ if for all $\epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap S \neq \emptyset$ and $(x-\epsilon, x+\epsilon) \cap C(S) \neq \emptyset$.

We point out that "int" is short for *interior* and "bd" is short for *boundary.*

B.13. For each S find $\text{int}(S)$ and $\text{bd}(S)$

(1) $[0,1]$
(2) $(0,1)$
(3) $\mathbb{Q}$
(4) $\mathbb{R}$
(5) $\{1,2,3\}$
(6) $\{\frac{1}{n} : n \in \mathbb{N}\}$

B.14. Suppose that $S \subseteq \mathbb{R}$. Prove the following

(1) $\text{int}(S) \subseteq S$ and $\text{int}(S)$ is an open set.
(2) S is open $\Leftrightarrow S = \text{int}(S)$.
(3) S is open $\Leftrightarrow S \cap \text{bd}(S) = \emptyset$.
(4) S is closed $\Leftrightarrow S \supseteq \text{bd}(S)$.

B.15. Prove the following

(1) If I is a set and for all $i \in I$, A_i is an open set, then $\bigcup_{i\in I} A_i$ is open.
(2) If I is any set and for all $i \in I$, F_i is a closed set then $\bigcap_{i\in I} F_i$ is closed.
(3) If $n \in \mathbb{N}$ and A_i is an open set for each $i \le n$ then $\bigcap_{i=1}^{n} A_i$ is open.
(4) If $n \in \mathbb{N}$ and A_i is a closed set for each $i \le n$ then $\bigcup_{i=1}^{n} A_i$ is closed.

B.16. Show by example that the last two parts of the previous problem cannot be extended to infinite intersections or unions.

Definition. Let $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

(1) x is an *accumulation point* of S if for all $\epsilon > 0$, $\{y \in \mathbb{R} : 0 < |x - y| < \epsilon\} \cap S \neq \emptyset$.
(2) $S' = \{x : x \text{ is an accumulation point of } S\}$.
(3) x is an *isolated point* of S if $x \in S \setminus S'$.
(4) $\bar{S} = S \cup S'$.

$\bar{S}$ is called the *closure* of S.

B.17. Let $S \subseteq \mathbb{R}$. Prove the following.

(1) $x \in S$ is an isolated point of S if and only if there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap S = \{x\}$. Let $S \subseteq \mathbb{R}$.
(2) Let $x \in \mathbb{R}$. Prove that $x \in S'$ if and only if for all $\epsilon > 0$, $(x - \epsilon, x + \epsilon) \cap S$ is infinite.

B.18. For each set S below find S', $\bar{S}$ and all isolated points of S.

(1) $\mathbb{R}$
(2) $\emptyset$
(3) $\mathbb{Q}$
(4) $(0, 1]$
(5) $\mathbb{Q} \cap (0, 1)$
(6) $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$
(7) $\{\frac{1}{n} : n \in \mathbb{N}\}$

B.19. Prove the following. $S \subseteq \mathbb{R}$.

(1) S is closed if and only if $S \supseteq S'$.
(2) $\bar{S}$ is closed.

(3) S is closed if and only if $S = \bar{S}$.
(4) If $F \supseteq S$ and F is closed then $F \supseteq \bar{S}$.

3. Compactness

Our next topic in topology is compactness. The definition is quite abstract and will take effort to absorb. We will later prove that a continuous function on a compact domain achieves both a maximum and a minimum value — quite a useful thing in applications.

Definition. Let $S \subseteq \mathbb{R}$.

(1) Let $\{A_i\}_{i \in I}$ be a family of open sets. $\{A_i\}_{i \in I}$ is an *open cover* for S if $S \subseteq \bigcup_{i \in I} A_i$.
(2) Let $\{A_i\}_{i \in I}$ be an open cover for S. A *subcover* of this open cover is any collection $\{A_i\}_{i \in I_0}$ where $I_0 \subseteq I$ such that $\bigcup_{i \in I_0} A_i \supseteq S$.
(3) S is *compact* if every open cover of S admits a finite subcover, i.e., whenever $\{A_i\}_{i \in I}$ is a family of open sets such that $S \subseteq \bigcup_{i \in I} A_i$ then there exists a finite set $F \subseteq I$ so that $S \subseteq \bigcup_{i \in F} A_i$.

For example, $\{(n-1, n+1) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$. For all $x \in \mathbb{Q}$, let ϵ_x be a positive number (which might be different for different x. Is $\{(x - \epsilon_x, x + \epsilon_x) : x \in \mathbb{Q}\}$ necessarily an open cover of $\mathbb{R}$?

This definition above is very abstract and may require study and time to absorb. Note that the definition requires that *every* open cover of S admits a finite subcover. To show S is not compact you only need construct *one* open cover without a finite subcover. Compactness plays a key role in analysis (and topology).

B.20. Which of the following sets are compact?

(1) $\{1, 2, 3\}$
(2) $\emptyset$
(3) $(0, 1)$
(4) $[0, 1)$
(5) $\mathbb{R}$

B.21. Let $S \subseteq \mathbb{R}$ be compact. Prove that

(1) S is bounded.
(2) S is closed.

Hint: Assume not in each case and produce an open cover without a finite subcover.

B.22. Prove that $[0, 1]$ is compact.

Hint: Let $\{A_i\}_{i \in I}$ be any open cover of $[0, 1]$. Let

$$B = \{x \in [0, 1] : [0, x] \text{ can be covered by a finite subcover of } \{A_i\}_{i \in I}\} .$$

Then $0 \in B$ so $B \neq \emptyset$. Let $x = \sup(B)$. Show $x \in B$. Show $x = 1$.

B.23. Let $K \subseteq \mathbb{R}$ be compact and let $F \subseteq K$ be closed. Prove that F is compact.

Hint: If $\{A_i\}_{i \in I}$ covers F then $\{A_i\}_{i \in I} \cup \{C(F)\}$ covers K.

B.24. Let $K \subseteq \mathbb{R}$ be closed and bounded.

(1) Prove $\min(K)$ and $\max(K)$ both exist if $K \neq \emptyset$.
(2) Prove that K is compact.

We see that $K \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ K is closed and bounded.

B.25. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a nested sequence of closed, bounded and nonempty sets in $\mathbb{R}$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Hint: Assume it is empty. Then

$$\mathbb{R} = C\left(\bigcap_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} C(I_n) \supseteq I_1 \ .$$

B.26. Let $K \subseteq \mathbb{R}$ be compact and infinite. Prove that $K' \neq \emptyset$.

Hint: Assume $K' = \emptyset$.

B.27. Let $A \subseteq \mathbb{R}$ be bounded and infinite. Prove that $A' \neq \emptyset$.

4. Sequential Limits and Closed Sets

Definition. Let $A \subseteq \mathbb{R}$. A is *sequentially* closed if whenever $(a_n)_{n=1}^{\infty}$ is a sequence in A converging to a limit a, then $a \in A$.

B.28. If $A \subseteq \mathbb{R}$ is closed then it is sequentially closed.

B.29. If $A \subseteq \mathbb{R}$ is sequentially closed then it is closed.

B.30. If $A \subseteq \mathbb{R}$ then A is closed if and only if it is sequentially closed.