M 365C

Fall 2013, Section 57465
Problem Set 10
Due Thu Nov 7

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-6, without reproving them.

Exercise 1 (*Rudin 6.2*)

Suppose $f(x) \geq 0$ for all $x \in [a,b]$, f is continuous, and $\int_a^b f(x)\,\mathrm{d}x = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Answer of exercise 1

Suppose for contradiction that $f(y) = c \neq 0$ at some $y \in [a,b]$. Then by continuity, there exists some neighborhood $N_\epsilon(y)$ such that $f(x) > c/2$ for all $x \in N$. Now choose a partition P of $[a,b]$ such that one of the intervals of the partition is $I = [y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2}]$. Let m be the infimum of $f(x)$ for $x \in I$; then $m \geq c/2$. The full lower sum $L(P,f)$ is obtained by summing the contribution from the interval I plus the contributions from other intervals. All those contributions are nonnegative, so $L(P,f)$ is at least the contribution from I, i.e. $L(P,f) \geq m\epsilon$. But then

$$\int_a^b f(x)\,dx \geq L(P,f) \geq m\epsilon > 0.$$

Exercise 2 (*Rudin 6.5*)

Suppose f is a bounded real function on $[a,b]$ and f^2 is Riemann integrable on $[a,b]$. Does it follow that f is Riemann integrable on $[a,b]$? Does the answer change if we assume instead that f^3 is Riemann integrable on $[a,b]$?

Answer of exercise 2

If f^2 is Riemann integrable it need not follow that f is; a counterexample is provided by the function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ -1 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

However, if f^3 is Riemann integrable then the situation is better. Indeed, for any x we can define a "cube root" $x^{1/3}$, such that $(x^3)^{1/3} = x$. (We had defined $x^{1/3}$ before only for $x \geq 0$; but we can extend it to $x < 0$ by defining $x^{1/3} = -|x|^{1/3}$ for $x < 0$. Then we can check directly that the resulting function indeed has $(x^3)^{1/3} = x$ for *all* x.) Moreover this function is continuous (we have proved before that it is continuous for $x \geq 0$, but this easily implies it is continuous for all x.) Then $f(x) = (f^3)^{1/3}$, and f^3 is integrable, so f is obtained by applying a continuous function to an integrable function, so f is also integrable.

Exercise 3 (*Rudin 6.7, in part*)

Suppose f is a real function on $(0,1]$ and f is Riemann integrable on $[c,1]$ for every $c>0$. We then define

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{c\to 0}\int_c^1 f(x)\,\mathrm{d}x$$

if this limit exists.

If f is Riemann integrable on $[0,1]$, show that this definition agrees with the old one.

Answer of exercise 3

The easy way: if f is Riemann integrable then the function $F(c)=\int_c^1 f(x)\mathrm{d}x$ is *continuous* on $[0,1]$ (using Rudin's Theorem 6.20). Thus

$$\lim_{c\to 0} F(c) = F(0)$$

which means

$$\lim_{c\to 0}\int_c^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$$

which is what we wanted to prove.

The harder way (doing it "by hand"): if f is Riemann integrable on $[0,1]$ then in particular it is bounded, say $|f(x)|<M$ for all $x\in[0,1]$. Thus

$$\left|\int_0^c f(x)\,\mathrm{d}x\right| \le \int_0^c |f(x)|\mathrm{d}x \le Mc$$

so

$$0 \le \liminf_{c\to 0}\left|\int_0^c f(x)\,\mathrm{d}x\right| \le \limsup_{c\to 0}\left|\int_0^c f(x)\,\mathrm{d}x\right| \le \lim_{c\to 0} Mc = 0$$

and hence

$$\lim_{c\to 0}\left|\int_0^c f(x)\,\mathrm{d}x\right| = 0$$

which is equivalent to

$$\lim_{c\to 0}\int_0^c f(x)\,\mathrm{d}x = 0.$$

Now

$$\int_c^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x - \int_0^c f(x)\,\mathrm{d}x$$

and so

$$\lim_{c\to 0}\int_c^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x - \lim_{c\to 0}\int_0^c f(x)\,\mathrm{d}x$$

i.e.

$$\lim_{c\to 0}\int_c^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$$

as desired.