M 365C

FALL 2013, SECTION 57465
PROBLEM SET 2
DUE THU SEP 12

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-2, without reproving them.

Exercise 1 (*Rudin 1.6*)

(This problem uses Theorem 1.21 of Rudin: for every real number $x > 0$ and integer $n > 0$, there exists a unique real number $x^{1/n}$ such that $x^{1/n} > 0$ and $(x^{1/n})^n = x$.)

Fix $b \in \mathbb{R}$, $b > 1$.

1. If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
$$(b^m)^{1/n} = (b^p)^{1/q}.$$
Hence it makes sense to define $b^r = (b^m)^{1/n}$. (Be sure you understand what this last sentence means!)
2. Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
3. If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
$$b^r = \sup B(r)$$
when r is rational.
4. By the result of the previous part, it makes sense to define
$$b^x = \sup B(x)$$
for every real x. With this definition, prove that $b^{x+y} = b^x b^y$ for every real x and y.

Answer of exercise 1

1. First note that $mq = pn$, so $b^{mq} = b^{pn}$. Thus
$$(b^{mq})^{1/nq} = (b^{pn})^{1/nq}.$$
What remains is to show that the left side of this is equal to $(b^m)^{1/n}$ and the right side is $(b^p)^{1/q}$. Both sides are similar, so let us just look at the left side. Let $\alpha = (b^m)^{1/n}$. By definition, α is the unique positive real number with $\alpha^n = b^m$. Thus $(\alpha^n)^q = (b^m)^q$. Using the general fact $(x^a)^b = x^{ab}$ when a, b are integers, which follows from the definition of exponentiation as repeated multiplication, we then have $\alpha^{nq} = b^{mq}$. But by definition $(b^{mq})^{1/nq}$ is the unique positive real number α obeying this equation. Thus $\alpha = (b^{mq})^{1/nq}$. So we have shown $(b^m)^{1/n} = (b^{mq})^{1/nq}$ as desired. This completes the proof.

2. Suppose we have two rational numbers $r = r_1/r_2$, $s = s_1/s_2$. Then $r + s = \frac{r_1 s_2 + r_2 s_1}{r_2 s_2}$ so $b^{r+s} = (b^{r_1 s_2 + r_2 s_1})^{1/r_2 s_2}$. Thus to show $b^r b^s = b^{r+s}$ it suffices to show $(b^r b^s)^{r_2 s_2} = b^{r_1 s_2 + r_2 s_1}$. But indeed $b^r b^s = ((b^{r_1})^{1/r_2}(b^{s_1})^{1/s_2})^{r_2 s_2} = b^{r_1 s_2} b^{r_2 s_1} = b^{r_1 s_2 + r_2 s_1}$ as desired.
3. First we show that for any $q \in \mathbb{Q}$ with $q > 0$, we have $b^q > 1$. To see this, write $q = m/n$ with $m, n > 0$. Then $(b^q)^n = b^m > 1$, from which it follows that $b^q > 1$ also (since for $n > 0$, $x^n > 1 \Leftrightarrow x > 1$).
   Now we show b^r is an upper bound for $B(r)$. For any $t \in B(r)$, i.e. $t \leq r$, we have $r - t \geq 0$ and thus $b^{r-t} \geq 1$, i.e. $b^r \leq b^t$.
   Finally we must show that if $x < b^r$ then x is not an upper bound for $B(r)$. But this is obvious since $b^r \in B(r)$.
4. First we show that $b^{x+y} \geq b^x b^y$. Assume the opposite, $b^{x+y} < b^x b^y$. Then $b^{x+y}/b^y < b^x$, so b^{x+y}/b^y is not an upper bound for $B(x)$. Thus there exists some $t < x$ such that $b^{x+y}/b^y < b^t$. Then $b^{x+y}/b^t < b^y$, so b^{x+y}/b^t is not an upper bound for $B(y)$. Thus there exists some $s < y$ such that $b^{x+y}/b^t < b^s$, i.e. $b^{x+y} < b^s b^t$. By the previous part, this means $b^{x+y} < b^{s+t}$. But $s + t < x + y$, so $b^{s+t} \in B(x + y)$. This contradicts the fact that b^{x+y} is an upper bound for $B(x + y)$.
   Next we show that $b^{x+y} \leq b^x b^y$. Assume the opposite, $b^{x+y} > b^x b^y$. Then $b^x b^y$ is not an upper bound for $B(x + y)$, so there exists some rational $t < x + y$ with $b^t > b^x b^y$. Now there exist rationals r and s with $r + s > t$, $r < x$, $s < y$. (To see this: choose any rational t' with $t < t' < x + y$. Then $t' - y < x$, so we may choose some rational r such that $t' - y < r < x$. Then let $s = t' - r$.) Now $b^r \leq b^x$, $b^s \leq b^y$, and hence $b^r b^s \leq b^x b^y$. By the previous part, this means $b^{r+s} \leq b^x b^y$. Thus $b^{r+s} < b^t$, so $b^{r+s-t} < 1$, but $r + s - t > 0$, which gives a contradiction.

Exercise 2 (*Rudin 2.2, modified*)

A real number x is called *algebraic* if there exists an $n \in \mathbb{N}$ and integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

Prove that the set of all algebraic real numbers is countable. (Hint: for every positive integer N, there are only finitely many ways to choose numbers $n, a_0, a_1, \ldots, a_n$ with $n + |a_0| + |a_1| + \cdots + |a_n| = N$.)

Answer of exercise 2

First we show that the set of all equations of the specified form is countable. Indeed, let E_n be the set of all equations of this form with n fixed. This set is just the set of all tuples $(a_0, \ldots, a_n) \in \mathbb{N}^n$; we have proven in class that this set is countable. Then the set of all equations of this form is $\cup_{n=1}^{\infty} E_n$, a countable union of countable sets, hence countable.

Next note that any polynomial equation of degree n has at most n distinct real solutions. (Strictly speaking, we should even prove this: one can do so directly, using the polynomial long division algorithm to see that for any polynomial $P(x)$ with $P(x) = 0$, one has $P(x) = (x - a)Q(x)$ where Q is a polynomial with $\deg Q = \deg P - 1$; then induction gives the desired statement.) Thus the set of all algebraic real numbers is contained in a countable union of finite sets, hence it is at most countable.

Finally, every integer is algebraic, so the set of all algebraic real numbers cannot be finite; hence it is countable.

Exercise 3 (*Rudin 2.5*)

Construct a bounded set of real numbers with exactly three limit points.

Answer of exercise 3

For any $x \in \mathbb{R}$, define $E_x = \{x + \frac{1}{n} \mid n \in \mathbb{N}\}$. Then for any real $a \neq b \neq c$, let $E = E_a \cup E_b \cup E_c$. E is evidently bounded and has limit points a, b, c.

Let us check that E has no other limit points. Thus suppose given a point y. If $y \notin \{a, b, c\}$ then take any ϵ such that $0 < \epsilon < \min\{|y - a|, |y - b|, |y - c|\}$. We claim $N_\epsilon(y)$ contains only finitely many points of E. Indeed, for $p = \frac{1}{n} + a$, we have $|y - p| \geq |y - a| - \frac{1}{n}$. But $|y - a| > \epsilon$. If $n > 1/(|y - a| - \epsilon)$ we will have $|y - p| \geq |y| - (|y| - \epsilon) = \epsilon$. Thus, $p \notin N_\epsilon(y)$ if $n > 1/(|y - a| - \epsilon)$. Thus $N_\epsilon(y) \cap E_a$ is finite, and similarly for $N_\epsilon(y) \cap E_b$ and $N_\epsilon(y) \cap E_c$. Thus $N_\epsilon(y) \cap E$ is finite.

Exercise 4 (*Rudin 2.6*)

Let E be a subset of a metric space X. Let E' be the set of all limit points of E. Let $\bar{E} = E \cup E'$ (the *closure* of E).

1. Prove that E' is closed.
2. Prove that $\bar{E}$ and E have the same limit points, i.e. $(\bar{E})' = E'$.
3. Do E and E' always have the same limit points? (If so, prove it; if not, give a counterexample.)

Answer of exercise 4

1. Suppose p is a limit point of E'; we want to show that $p \in E'$, i.e. that p is a limit point of E. So, let U be any neighborhood of p. U contains some point $q \in E'$, $q \neq p$. Now, since U is open and $q \in U$, we can find some neighborhood V of q which is contained in U. Since q is a limit point of E, V contains some point r of E. Thus $r \in V \subset U$, so U contains a point of E; but U was an arbitrary neighborhood of p, so we conclude p is a limit point of E.
2. Obviously any limit point of E is also a limit point of $\bar{E}$. It remains to prove that any limit point of $\bar{E}$ is a limit point of E. Thus, suppose p is a limit point of $\bar{E}$. Then take any neighborhood U of p. U contains some point $q \in \bar{E}$. Since U is open we can find a neighborhood V of q, $V \subset U$. V must contain some point r of E (if $q \in E$ then we can simply take $r = q$; otherwise $q \in E'$ and then V contains some $r \neq q$ with $r \in E$.) This $r \in V \subset U$, so U contains a point of E; but U was an arbitrary neighborhood of p, so we conclude p is a limit point of E.

3. No. For example we can take $E = \{1/n \mid n \in \mathbb{N}\}$, which has $E' = \{0\}$. Then $E'' = \emptyset$, so $E' \neq E''$.

Exercise 5 (*Rudin 2.9, in part*)

Let E be a subset of a metric space X. Let E° be the set of all interior points of E.

1. If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
2. Do E and $\bar{E}$ always have the same interior, i.e. does $E^\circ = (\bar{E})^\circ$? (If so, prove it; if not, give a counterexample.)

Answer of exercise 5

1. Fix some $p \in G$. Since G is open, there exists a neighborhood U of p with $U \subset G$. But $G \subset E$, so also $U \subset E$. Thus p has a neighborhood contained in E, i.e. p is an interior point of E.
2. No. For example, we can take $E = (-1, 0) \cup (0, 1)$. This set is open and thus $E^\circ = E$. On the other hand $\bar{E} = [-1, 1]$ which has interior $(\bar{E})^\circ = (-1, 1)$.

Exercise 6 (*Rudin 2.10, in part*)

Let X be any set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases} \tag{1}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed?

Answer of exercise 6

To see that d is a metric the only nontrivial point is to verify the triangle inequality $d(p, q) \leq d(p, r) + d(q, r)$. If $p = q$ then the left side is zero while the right side is nonnegative, so in this case the inequality is satisfied. If $p \neq q$ then the left side is 1 and at least one of the terms on the right side is 1, while the other is nonnegative, so again the inequality is satisfied.

Now let E be any subset of X, and consider any $p \in E$. The neighborhood $N_{1/2}(p) = \{p\}$ (since every point $q \neq p$ has $d(p, q) = 1 > 1/2$.) Thus $N_{1/2}(p) \subset E$, and hence p is an interior point of E. It follows that E is open. So every subset of the metric space X is open.

Finally, let E be any subset of X again; then E^c is open (since every subset of X is open); thus E is closed. So every subset of the metric space X is closed.