M 365C

FALL 2013, SECTION 57465
PROBLEM SET 5
DUE THU OCT 3

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-3, without reproving them.

Exercise 1

Let X be a metric space, $\{p_n\} \subset X$ a convergent sequence with $p_n \to p$, and $\{q_n\} \subset X$ a convergent sequence with $q_n \to q$. Prove that $d(p_n, q_n) \to d(p, q)$. (This last convergence takes place in $\mathbb{R}$.)

Answer of exercise 1

Fix some $\epsilon > 0$. Then there exists some N' such that $n > N' \implies d(p_n, p) < \epsilon/2$, and there exists some N'' such that $n > N'' \implies d(q_n, q) < \epsilon/2$. Let $N = \max(N', N'')$. Using the triangle inequality in $\mathbb{R}$, we have

$$|d(p_n, q_n) - d(p, q)| \le |d(p_n, q_n) - d(p_n, q)| + |d(p_n, q) - d(p, q)|$$

Next we can use the triangle inequality in X, to get

$$|d(p_n, q_n) - d(p_n, q)| \le d(q, q_n)$$

and

$$|d(p_n, q) - d(p, q)| \le d(p, p_n)$$

Combining these, for $n > N$ we have

$$|d(p_n, q_n) - d(p, q)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus $d(p_n, q_n) \to d(p, q)$ as desired.

Exercise 2 (*Rudin 3.2, modified*)

Calculate $\lim_{n\to\infty} \sqrt{n^2+n} - n$, and prove that your answer is correct. (Hint: first show that $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$.)

Answer of exercise 2

Multiplying out shows directly that $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$. Now, dividing by n in numerator and denominator, this becomes $\frac{1}{\sqrt{1+1/n}+1}$. We will prove below that $\sqrt{1+1/n} \to 1$; having proved that, it will follow that the desired limit is $1/2$.

We want to show that $\sqrt{1+1/n} \to 1$. So, let $a_n = \sqrt{1+1/n}$. We have $a_n^2 - 1 = 1/n$. Factoring, this becomes $(a_n+1)(a_n-1) = 1/n$, i.e. $a_n - 1 = \frac{1}{n(a_n+1)}$. Since $a_n > 0$ this says

$a_n - 1 < 1/n$. On the other hand we can easily see that $a_n > 1$. Thus $1 < a_n < 1 + 1/n$, so $|a_n - 1| < 1/n$. Thus, for any ϵ, if we choose $N > 1/\epsilon$, then for all $n \leq N$ we have $|a_n - 1| < \epsilon$. Thus $a_n \to 1$ as desired.

Exercise 3 (*Rudin 3.5*)

For any two real sequences $\{a_n\}$, $\{b_n\}$ prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.$$

Answer of exercise 3

Let $\gamma = \lim_{n\to\infty} \sup(a_n + b_n)$, $\alpha = \lim_{n\to\infty} \sup a_n$, $\beta = \lim_{n\to\infty} \sup b_n$. Fix some $\epsilon > 0$.

By the definition of γ, there exists a subsequence $\{a_{k_n} + b_{k_n}\}$ which converges to a limit greater than $\gamma - \epsilon/3$. Then there exists some N such that $n > N$ implies $a_{k_n} + b_{k_n} > \gamma - \epsilon/3$. Also by the definition of α and β, there exists some N' such that $n > N'$ implies $a_{k_n} < \alpha + \epsilon/3$, and some N'' such that $n > N''$ implies $b_{k_n} < \beta + \epsilon/3$. Thus if we take $n > \max(N, N', N'')$ we will have

$$\gamma - \epsilon/3 < a_{k_n} + b_{k_n} < \alpha + \beta + 2\epsilon/3$$

and thus $\gamma < \alpha + \beta + \epsilon$, for any $\epsilon > 0$. Thus $\gamma \leq \alpha + \beta$ as desired.