M 365C
Fall 2013, Section 57465
Problem Set 6
Due Thu Oct 10

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-3, without reproving them.

Exercise 1 (*Rudin 3.6, modified*)

Investigate the behavior (convergence or divergence) of $\sum a_n$ if

1. $a_n = \sqrt{n+1} - \sqrt{n}$,
2. $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$,
3. $a_n = (n^{1/n} - 1)^n$.

Answer of exercise 1

1. The partial sums are just $s_n = \sum_{k=1}^n a_k = \sqrt{n+1} - 1$. But for any $M \in \mathbb{R}$, if we take $n > (M+1)^2 - 1$, then $\sqrt{n+1} - 1 > M$. Thus s_n is unbounded above, and thus $\lim_{n\to\infty} s_n$ does not exist. Thus $\sum a_n$ diverges.
2. In a previous assignment you have shown that $\sqrt{n}(\sqrt{n-1} - \sqrt{n}) \to \frac{1}{2}$. It follows that there exists N such that for all $n \geq N$, $\sqrt{n}(\sqrt{n-1} - \sqrt{n}) < 1$. Thus for all $n \geq N$, $\sqrt{n-1} - \sqrt{n} < \frac{1}{\sqrt{n}}$, so $0 < a_n < \frac{1}{n^{3/2}}$. But $\sum_n \frac{1}{n^{3/2}}$ converges. It follows that $\sum a_n$ also converges.
3. Apply the root test: let $\alpha = \lim_{n\to\infty} \sup(n^{1/n} - 1)$. Rudin shows that $n^{1/n} \to 1$. Thus $\alpha = 1 - 1 = 0 < 1$, so $\sum_n a_n$ converges.

Exercise 2 (*Rudin 3.8*)

If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Answer of exercise 2

This problem is deceptively tricky. It is tempting to try to do it by establishing some inequality like $|\sum_{n=p}^q a_n b_n| \leq M|\sum_{n=p}^q a_n|$, but I think this won't work: no such inequality can exist, since in some cases we might have $|\sum_{n=p}^q a_n| = 0$ but $\sum_{n=p}^q a_n b_n \neq 0$. (for example imagine that $a_1 = 1$, $a_2 = -1$, while $b_1 = 1$, $b_2 = 1/2$, and consider $p = 1$, $q = 2$.) In fact, it is pretty delicate to get the estimate we want, but fortunately Rudin has done the hard work for us, as follows.

$\{b_n\}$ is monotonic and bounded, so it has some limit M. Let $c_n = b_n - M$. Then $\{c_n\}$ is also monotonic and $c_n \to 0$. Since $a_n b_n = a_n(c_n + M)$, and $\sum M a_n$ converges, we see that $\sum a_n b_n$ converges if and only if $\sum a_n c_n$ converges. If $\{c_n\}$ is monotonically *decreasing* then Theorem 3.42 of Rudin shows that $\sum a_n c_n$ indeed converges. If $\{c_n\}$ is monotonically *increasing* then we define $c'_n = -c_n$ and apply Theorem 3.42 to $\sum a_n c'_n$.

Exercise 3 (*Rudin 3.20*)

Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_k}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Answer of exercise 3

Fix $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that $n > N \implies d(p_n, p) < \epsilon$. Since $\{p_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $m, n > N \implies d(p_m, p_n) < \epsilon/2$. Since the subsequence $\{p_{n_k}\}$ converges to p, there exists some $k \in \mathbb{N}$ such that $d(p_{n_k}, p) < \epsilon/2$ and $n_k > N$. Then for $n > N$ we have

$$d(p_n, p) < d(p_n, p_{n_k}) + d(p_{n_k}, p) = \epsilon/2 + \epsilon/2 = \epsilon.$$

Exercise 4 (*Rudin 3.21*)

Suppose $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, with $E_n \supset E_{n+1}$, and $\lim_{n\to\infty} \operatorname{diam} E_n = 0$. Prove that $\cap_{n=1}^{\infty} E_n$ consists of exactly one point.

Answer of exercise 4

Pick a sequence $\{p_n\}$ in X, with $p_n \in E_n$ for all n. We will show that $\{p_n\}$ is a Cauchy sequence. Indeed, fix some $\epsilon > 0$. Since $\operatorname{diam} E_n \to 0$, there exists some $N \in \mathbb{N}$ for which $\operatorname{diam} E_N < \epsilon$. For any $n, m > N$ we have $p_n, p_m \in E_N$ and thus $d(p_n, p_m) < \epsilon$, so $\{p_n\}$ is a Cauchy sequence as desired. Since X is complete, it follows that $\{p_n\}$ converges; let p be the limit.

Now we want to show that $p \in E_N$ for all N. Since $p_n \to p$, for any $\epsilon > 0$ there is some n for which $p_n \in N_\epsilon(p)$; also $p_n \in E_N$, so $N_\epsilon(p)$ contains a point of E_N. Hence p is a limit point of E_N. But E_N is closed, so this means $p \in E_N$. Since this holds for all N, we have $p \in \cap_{n=1}^{\infty} E_n$.

Finally, we need to show that p is the *only* point in $\cap_{n=1}^{\infty} E_n$. So, suppose there is some $q \in \cap_{n=1}^{\infty} E_n$ with $q \neq p$. Then there exists some n such that $\operatorname{diam}(E_n) < d(p, q)$. Since both p and q are in E_n, this is a contradiction.

* Exercise 5 (*Rudin 3.22*)

Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of open subsets of X, such that each G_n is dense in X. Prove that $\cap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X.) (Hint: find a shrinking sequence of neighborhoods E_n such that $\bar{E}_n \subset G_n$, and apply the previous exercise.)

Answer of exercise 5

It is useful to first think about an easier question: how do we know even that $G_1 \cap G_2$ is not empty? Pick any $q \in G_1$. Since G_1 is open, there is some neighborhood N of q with $N \subset G_1$. Then since G_2 is dense in X, either $q \in G_2$ or q is a limit point of G_2; in either case N contains a point $p \in G_2$. Then $p \in G_1 \cap G_2$.

Now we consider the real question.

First, let p_1 be any point of G_1. Since G_1 is open, there exists some ϵ for which $N_\epsilon(p_1) \subset G_1$. Picking some $\epsilon' < \epsilon$, let $E_1 = N_{\epsilon'}(p_1)$; then $\bar{E}_1 \subset N_\epsilon(p_1) \subset G_1$.

Next, take any $q_2 \in E_1$. Since E_1 is open, there is some neighborhood N of q_2 such that $N \subset E_1$. Then since G_2 is dense in X, either $q_2 \in G_2$ or q_2 is a limit point of G_2; in either case N contains a point $p_2 \in G_2$. Now, since N and G_2 are both open, there exists some ϵ such that $N_\epsilon(p_2) \subset N \subset E_1$ and also $N_\epsilon(p_2) \subset G_2$. Pick some $\epsilon' < \epsilon$, and let $E_2 = N_{\epsilon'}(p_2)$; then $\bar{E}_2 \subset N_\epsilon(p_2) \subset E_1$ and also $\bar{E}_2 \subset G_2$.

Continuing in this way we obtain subsets $E_1 \supset E_2 \supset E_3 \supset \cdots$ with $\bar{E}_n \subset G_n$. By shrinking E_n if necessary at each step, we can arrange that $\operatorname{diam} E_n < 1/n$, so $\operatorname{diam} E_n \to 0$. Thus, using the previous exercise, $\cap_{n=1}^\infty \bar{E}_n$ contains a single point p. Then p is also in $\cap_{n=1}^\infty G_n$.