M 365C

Fall 2013, Section 57465
Problem Set 7
Due Thu Oct 17

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-4, without reproving them.

Exercise 1 (*Rudin 3.10*)

Suppose that the coefficients a_n of the power series $\sum a_n x^n$ are integers, infinitely many of which are nonzero. Prove that the radius of convergence of the series is at most 1.

Answer of exercise 1

Since infinitely many $|a_n| \geq 1$, it follows that infinitely many $|a_n|^{1/n} \geq 1$. But then

$$\lim_{n\to\infty} \sup |a_n|^{1/n} \geq \lim_{n\to\infty} \sup 1 = 1.$$

By the root test the desired result follows.

Exercise 2

1. Consider the sequence of functions $\{f_n\}$ where $f_n : [0,1] \to [0,1]$ is given by $f_n(x) = x^n$. Show that for each $x \in [0,1]$ the sequence of real numbers $\{f_n(x)\}$ converges, and compute the limit.
2. Let $f(x)$ denote the limit you computed in the previous part; this gives a new function $f : [0,1] \to \mathbb{R}$. Is f continuous?

Answer of exercise 2

We have shown in class that $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$; and $\lim_{n\to\infty} x^n = \lim_{n\to\infty} 1 = 1$ for $x = 1$. Thus

$$f(x) = \begin{cases} 0 \text{ if } x \in [0,1), \\ 1 \text{ if } x = 1. \end{cases}$$

This function is not continuous at $x = 1$: for example, taking $\epsilon = 1/2$, there is evidently no δ such that $|x - 1| < \delta \implies |f(x) - 1| < \epsilon$.

Exercise 3

Let X be any set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

With this distance function, X is a metric space (as you proved in an earlier homework). Let Y be any metric space. Prove that every function $f : X \to Y$ is continuous.

Answer of exercise 3

For any $\epsilon > 0$, set $\delta = 1/2$. Then $d(p,q) < \delta \implies p = q$, which in turn implies $f(p) = f(q)$, and thus $d(f(p), f(q)) < \epsilon$, as needed.

Exercise 4 (*Rudin 4.1*)

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h\to 0} (f(x+h) - f(x-h)) = 0$$

for all $x \in \mathbb{R}$. Does it follow that f is continuous?

Answer of exercise 4

Consider the function

$$f(x) = \begin{cases} 0 \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

This function is not continuous at $x = 0$ (for example, taking $\epsilon = 1/2$, there is evidently no δ such that $|x - 0| < \delta \implies |f(x) - 1| < \epsilon$.) However, $f(x)$ does obey the equation $\lim_{h\to 0} (f(x+h) - f(x-h)) = 0$. Indeed, for any x we have $\lim_{h\to 0} f(x+h) = 0$ and $\lim_{h\to 0} f(x-h) = 0$. (To prove this, for any ϵ, when $x \neq 0$ take any $\delta < |x|$ in the definition of limit, and when $x = 0$ take any $\delta > 0$.)

Thus the answer to the question is "no, it does not follow from this equation that f is continuous."

Exercise 5 (*Rudin 4.2*)

If X, Y are metric spaces and $f : X \to Y$ is continuous, prove that for every $E \subset X$,

$$f(\bar{E}) \subset \overline{f(E)}.$$

Prove by an example that this inclusion may be proper, i.e. it may happen that $f(\bar{E}) \neq \overline{f(E)}$.

Answer of exercise 5

Consider any point $y \in f(\bar{E})$. If $y \in f(E)$ then obviously $y \in \overline{f(E)}$. If $y \notin f(E)$, then $y = f(x)$ for x a limit point of E. In this case we may choose a sequence $\{x_n\}$ in E with $x_n \to x$, $x_n \neq x$. Then since f is continuous we also have $f(x_n) \to f(x)$. This says that $f(x)$ is a limit point of $f(E)$. But $f(x) = y$, so y is a limit point of $f(E)$, so $y \in \overline{f(E)}$.

To see that the inclusion may be proper, take $X = \mathbb{R}_+$, $Y = \mathbb{R}$, $E = \mathbb{N} \subset \mathbb{R}_+$, and let $f : \mathbb{R}_+ \to \mathbb{R}$ be given by $f(x) = 1/x$. Then $\bar{E} = \mathbb{N}$ so $f(\bar{E}) = \{1/n \,|\, n \in \mathbb{N}\}$, while $\overline{f(E)} = \{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$.

Exercise 6 (*Rudin 4.3*)

Let X be a metric space and $f : X \to \mathbb{R}$ be continuous. Let $Z(f)$ be the set of all $p \in X$ for which $f(p) = 0$. Prove that $Z(f)$ is closed in X.

Answer of exercise 6

By definition $Z(f) = f^{-1}(\{0\})$. $\{0\}$ is a finite set, hence a closed subset of $\mathbb{R}$. Since f is continuous, it follows that $Z(f)$ is closed.