M 365C

Fall 2013, Section 57465
Problem Set 8
Due Thu Oct 24

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-4, without reproving them.

Exercise 1

1. For any $k \in \mathbb{N}$, show that the function $f : [0,1] \to [0,1]$ defined by $f(x) = x^{1/k}$ is continuous. (Hint: you could do this directly from the definition of continuity, but there is an easier way.)
2. Let $a_n = (1 - 1/n)^{1/3}$. Show that $a_n \to 1$.
3. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$. For any $k \in \mathbb{N}$, show that the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(x) = x^{1/k}$ is continuous. (Hint: one approach is to reduce this to part 1 above.)

Answer of exercise 1

1. The map $g : [0,1] \to [0,1]$ defined by $g(x) = x^k$ is a continuous bijection whose domain is the compact set $[0,1]$. It follows that the inverse g^{-1} is also continuous. But $g^{-1}(x) = x^{1/k} = f(x)$. So f is continuous.
2. Let $b_n = 1 - 1/n$. Each $b_n \in [0,1]$, $b_n \to 1$ and $f(b_n) = a_n$. Since f is continuous it follows that $f(b_n) \to f(1) = 1^{1/k} = 1$. Thus $a_n \to 1$ as desired.
3. First note that the same method we used in part 1 would equally show that the function $f : [0,c] \to [0,c^{1/k}]$ defined by $f(x) = x^{1/k}$ is continuous, for any $c > 0$. Now, any $x \in \mathbb{R}_+$ is contained in $[0,c]$ for sufficiently large c. This looks like it should be sufficient to show that $f : \mathbb{R}_+ \to \mathbb{R}$ defined by the same formula is continuous. However, if we are really careful, we might notice that a small lemma is still missing. We state and prove the needed lemma as part of the solution to exercise 2 below.

Exercise 2

Prove that continuity is a local property, in the following sense. Let X and Y be metric spaces, and fix some $p \in X$. Suppose given two functions $f, g : X \to Y$. Suppose that f is continuous at p, and there exists a neighborhood N of p such that $f(q) = g(q)$ for all $q \in N$. Then, prove that g is continuous at p.

Answer of exercise 2

We will prove something slightly more general. Suppose X and Y are metric spaces, with $E \subset X$, and fix some $p \in E$. Suppose given two functions $f : E \to Y$ and $g : X \to Y$. Suppose that f is continuous at p, and there exists a neighborhood N of p, with $N \subset E$, such that $f(q) = g(q)$ for all $q \in N$. Then, we will prove that g is continuous at p.

The proof is as follows. Suppose given any sequence p_n in X with $p_n \to p$. We need to show that $g(p_n) \to g(p)$. For sufficiently large n we have $p_n \in N$. We may as well assume that all $p_n \in N$ (since throwing away finitely many terms from the sequence $\{g(p_n)\}$ does not affect the limit.) Since f is continuous at p, it follows that $f(p_n) \to f(p)$. Since $f(p_n) = g(p_n)$ and $f(p) = g(p)$ it follows that $g(p_n) \to g(p)$ as desired.

Exercise 3 (*Rudin 4.3, modified*)

Let X and Y be any metric spaces. Let $f : X \to Y$ and $g : X \to Y$ be continuous. Let E be a dense subset of X, such that $f(p) = g(p)$ for all $p \in E$. Then, show that $f(p) = g(p)$ for all $p \in X$. (So, a continuous function can be determined by its values on a dense set.)

Answer of exercise 3

Take any $p \in X$. Since $p \in \bar{E}$, there exists some sequence $\{p_n\}$ in E with $p_n \to p$. Then, since f and g are continuous, we have $f(p_n) \to f(p)$ and $g(p_n) \to g(p)$. But since all $p_n \in E$, $\{f(p_n)\}$ and $\{g(p_n)\}$ are the same sequence. Thus by uniqueness of the limits of sequences, $f(p) = g(p)$.

Exercise 4 (*Rudin 4.14*)

Let $I = [0, 1]$. Suppose $f : I \to I$ is continuous. Prove that there exists some $x \in I$ for which $f(x) = x$.

Answer of exercise 4

If $f(0) = 0$ or $f(1) = 1$ then we are done.

Otherwise $f(0) > 0$ and $f(1) < 1$. Consider the function $g(x) = f(x) - x$. $g(x)$ is continuous, since $f(x)$ and x are, and $g(x) > 0$, $g(1) < 0$. Thus by the intermediate-value theorem we have $g(x) = 0$ for some $x \in (0, 1)$. But this says $f(x) - x = 0$, i.e. $f(x) = x$, as needed.

Exercise 5

Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

Answer of exercise 5

Take $\epsilon = 1$, and take any $\delta > 0$. Set $x_1 = 1/\delta + \delta/2$, $x_2 = 1/\delta$. Then

$$|x_1^2 - x_2^2| = 1 + \delta^2/4 > 1 = \epsilon.$$

Thus there is no δ satisfying the definition of uniform continuity.

* Exercise 6 (*Rudin 4.8, modified*)

1. Let $E \subset \mathbb{R}$ be bounded. Let $f : E \to \mathbb{R}$ be uniformly continuous. Prove that $f(E) \subset \mathbb{R}$ is bounded.
2. Give a counterexample to the above if we omit the word "uniformly."

Answer of exercise 6

1. Take $\epsilon = 1$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < 1$.
   Since E is bounded, there is some interval $[a, b]$ such that $E \subset [a, b]$. Thus we may cover E by a finite number of closed intervals $I_1, \ldots, I_n$ with lengths at most $\delta/2$. Each $f(I_k)$ is bounded since $x, y \in I_k \implies |f(x) - f(y)| < 1$. But then $f(E)$ is the union of finitely many bounded sets, hence also bounded.
2. Take $E = (0, 1)$ and $f(x) = 1/x$.