M 365C

Fall 2013, Section 57465

Problem Set 9

Due Thu Oct 31

In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-5, without reproving them.

Exercise 1 (*Rudin 5.1*)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ and for all $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \leq (x - y)^2.$$

Prove that f is constant.

Answer of exercise 1

We can rewrite the given equation as $|f(x)-f(y)| \leq |x-y|^2$. We have $\frac{|f(x)-f(y)|}{|x-y|} < |x-y|$, i.e. $\left|\frac{f(x)-f(y)}{x-y}\right| < |x-y|$. Thus, by the definition of limit, $\lim_{x\to y} \frac{f(x)-f(y)}{x-y} = 0$. But this means f is differentiable at y and $f'(y) = 0$. This holds for all y. Thus (by a theorem proven in class) f is constant.

Exercise 2 (*Rudin 5.2*)

Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and has $f'(x) > 0$ for all $x \in (a, b)$.

1. Prove that f is strictly increasing, i.e. if $y > x$ then $f(y) > f(x)$.
2. Prove that the image of f is an interval (c, d) (the values $c = -\infty$ and $d = +\infty$ are allowed.)
3. Prove that $f : (a, b) \to (c, d)$ is bijective. Thus f has an inverse $g : (c, d) \to (a, b)$.
4. Prove that g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$ for all $x \in (a, b)$.

Answer of exercise 2

1. This follows directly from the mean value theorem: for $y > x$ with both y and x in (a, b) we have $f(y) - f(x) = (y - x)f'(c)$ for some c also in (a, b), and thus $f(y) - f(x) > 0$.
2. Let E be the image of f. Let $c = \inf E$ and $d = \sup E$. Then E does not contain any $x < c$ or any $x > d$. Also E does not contain c or d. Indeed, if E contains c then there is some $t \in (a, b)$ for which $f(t) = c$, but then taking $t' \in (a, b)$ with $t' < t$ it would follow that $f(t') < f(t)$, i.e. $f(t') < c$, contradicting the fact that E does not contain any $x < c$; a similar argument shows E does not contain d.
   On the other hand, by the definition of inf, for any $z \in (c, d)$, E does contain an element $x < z$, and by the definition of sup, E also contains an element $y > z$. Next, since f is continuous, we know that E is connected; thus, since it is a connected

subset of $\mathbb{R}$, E has the property that if $x, y \in E$ and $x < z < y$ then $z \in f(E)$ also. Thus $z \in (c, d)$.
We have now shown that any $z \notin (c, d)$ is not in E, and any $z \in (c, d)$ is in E, so $E = (c, d)$.

3. We already know f is surjective onto (c, d), so we just need to know it is injective; but this is immediate since if $y > x$ we know $f(y) > f(x)$, in particular $f(y) \neq f(x)$.
4. First a remark: if we already knew that g is differentiable then this would follow immediately from the chain rule, using the fact that $x = g(f(x))$. Here we don't know in advance that g is differentiable, but we do know that the composition $g(f(x))$ is differentiable. So we essentially need a very special case of the computation which appears in the proof of the chain rule. So, consider any sequence t_n with $t_n \to f(x)$. Since f is bijective, each $t_n = f(u_n)$ for some u_n. Also, g is continuous (to see this, restrict to a small compact domain around x and then use the fact that the inverse of continuous function on compact domain is continuous). Thus $g(t_n) \to g(f(u_n))$, i.e. $u_n \to x$. Then
$$\frac{g(f(x)) - g(t_n)}{f(x) - t_n} = \frac{x - u_n}{f(x) - f(u_n)} \to \frac{1}{f'(x)}.$$
But we have this for every sequence $t_n \to f(x)$, so
$$\lim_{t \to f(x)} \frac{g(f(x)) - g(t)}{f(x) - t} = \frac{1}{f'(x)}$$
which gives $g'(f(x))$ as desired.

Exercise 3 (*Rudin 5.3*)

Suppose that $g : \mathbb{R} \to \mathbb{R}$ and that there exists some M such that for all $x \in \mathbb{R}$, $|g'(x)| \leq M$. For $\epsilon > 0$, define $f_\epsilon(x) = x + \epsilon g(x)$. Prove that f_ϵ is 1-1 if ϵ is small enough, i.e. show that there is some $\epsilon' > 0$ such that, if $0 < \epsilon < \epsilon'$, then f_ϵ is 1-1.

Answer of exercise 3

By the mean value theorem, f_ϵ will be 1-1 if we have $f'_\epsilon(x) > 0$ for all x. But $f'_\epsilon(x) = 1 + \epsilon g'(x) \geq 1 - \epsilon M$ for all x, so if $\epsilon < 1/M$ we will have $f'_\epsilon(x) > 0$ for all x.

Exercise 4 (*Rudin 5.11*)

Suppose $f : (a, b) \to \mathbb{R}$ is twice differentiable at x. Show that
$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$
(Hint: use L'Hospital's rule, Theorem 5.13.)

Answer of exercise 4

As $h \to 0$ the numerator and denominator both approach 0. Thus we may apply L'Hospital's rule, differentiating both top and bottom with respect to h to see that the limit we are interested in is equal to

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

if the latter limit exists. The latter limit is a "symmmetrized" version of the usual limit defining the second derivative. To see that it really equals the usual second derivative, perhaps the fastest way is to note that we can rewrite it as

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} + \lim_{h \to 0} \frac{f'(x) - f'(x-h)}{h}$$

if those two limits separately exist. Fortunately they do exist and are both equal to $\frac{1}{2}f''(x)$ (now just using the definition of derivative), so finally we get $f''(x)$ as desired.

Exercise 5 (*Rudin 4.7*)

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} xy^2/(x^2+y^4) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. Given any $E \subset \mathbb{R}^2$, let $f|_E$ denote the *restriction* of f to E, i.e. $f|_E : E \to \mathbb{R}$ is defined by $f|_E(x) = f(x)$ for all $x \in E$. If E is a straight line through $(0,0)$ in $\mathbb{R}^2$, prove that $f|_E$ is continuous at $(x,y) = (0,0)$.
2. Prove nevertheless that f is not continuous at $(x,y) = (0,0)$!

Answer of exercise 5

1. Along the line $x = 0$ we have $f(x,y) = 0$, so f is evidently continuous when restricted to this line. Along any other line E we can write $y = cx$ for some constant c. Now consider any sequence of points (x_n, y_n) which lie in E and approach $(0,0)$. Then in particular $x_n \to 0$. Now

$$f(x_n, y_n) = f(x_n, cx_n) = \frac{c^2 x_n^3}{c^2 x_n^2 + c^4 x_n^4} = \frac{c^2 x_n}{c^2 + c^4 x_n^2} \to 0$$

   as $x_n \to 0$. And $f(0,0) = 0$. So we have shown that if $(x_n, y_n) \to (0,0)$ and $(x_n, y_n) \in E$ then $\lim_{n\to\infty} f(x_n, y_n) = f(0,0)$. It follows that $f|_E$ is continuous at $(0,0)$.
2. Now consider a sequence (x_n, y_n) with $y_n \to 0$ and $x_n = y_n^2$. Then $(x_n, y_n) \to (0,0)$. But

$$f(x_n, y_n) = \frac{y_n^4}{2y_n^4} = \frac{1}{2}$$

   so $f(x_n, y_n) \to \frac{1}{2}$, while $f(0,0) = 0$. Thus f is not continuous at $(0,0)$.