Differentiation

Def If $f:[a,b]\to\mathbb{R}$, and $x\in[a,b]$, consider

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

If this limit exists, we call it $f'(x)$, and say f is <u>differentiable</u> at x.

Thus, let $E=\{x\in[a,b] \mid f \text{ diff'ble at } x\}$; $f':E\to\mathbb{R}$ is the <u>derivative</u> of f.

Thm $f:[a,b]\to\mathbb{R}$ diff'ble at $x \Rightarrow f$ continuous at x.

Pf $\lim_{t\to x} f(t)-f(x) = f'(x)\cdot\lim_{t\to x} t-x = 0$. Thus $\lim_{t\to x} f(t)=f(x)$. ∎

Thm $f,g:[a,b]\to\mathbb{R}$ diff'ble at $x\Rightarrow$

1) $f+g$ diff'ble at x, $(f+g)'(x)=f'(x)+g'(x)$
2) fg diff'ble at x, $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$

Pf 1) easy.

2)
$$\lim_{t\to x}\frac{(fg)(t)-(fg)(x)}{t-x} = \lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)}{t-x}$$
$$=\lim_{t\to x}\frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x}$$
$$=\lim_{t\to x} g(t)\cdot\frac{f(t)-f(x)}{t-x}+f(x)\cdot\frac{g(t)-g(x)}{t-x}$$
$$= g(x)f'(x) + f(x)g'(x)$$

Cor For $n\in\mathbb{N}$, $(x^n)'=nx^{n-1}$

Pf Induction on n: for $n=0$, $(1)'=0$ ✓

for $n>0$, $(x^n)'=(x\cdot x^{n-1})'=1\cdot x^{n-1}+x\cdot(n-1)x^{n-2}=nx^{n-1}$ ∎

Prop $\left(\frac{1}{x}\right)'=-\frac{1}{x^2}$

Pf $\lim_{t\to x}\frac{\frac{1}{t}-\frac{1}{x}}{t-x}=\lim_{t\to x}\frac{\left(\frac{x-t}{tx}\right)}{t-x}=\lim_{t\to x}\left(-\frac{1}{xt}\right)=-\frac{1}{x^2}$

Thm If:

f cts on $[a,b]$ and diff'ble at $x \in [a,b]$

g defined on an interval I containing $f([a,b])$

g diff'ble at $f(x)$

$h(t) = g(f(t)) \qquad t \in [a,b]$

Then: h is diff'ble at x, $h'(x) = g'(f(x))\, f'(x)$

Pf Say $y = f(x)$.

Have $f(t) - f(x) = (t-x)(f'(x) + u(t)) \qquad t \in [a,b] \qquad u(t) \to 0$ as $t \to x$

$g(s) - g(y) = (s-y)(g'(y) + v(s)) \qquad s \in I \qquad v(s) \to 0$ as $s \to y$

Say $s = f(t)$.

$$h(t) - h(x) = g(f(t)) - g(f(x))$$
$$= (f(t) - f(x))(g'(y) + v(s))$$
$$= (t-x)(f'(x) + u(t))(g'(y) + v(s))$$

and $\lim_{t\to x} u(t) = 0$, $\lim_{t \to x} v(f(t)) = 0$ [take any $\{t_n\}$ with $t_n \to x$, then $f(t_n) \to y$ since f continuous; so $v(f(t_n)) \longrightarrow \lim_{s\to y} v(s) = 0$]

so $\lim_{t\to x} \dfrac{h(t)-h(x)}{t-x} = f'(x)\, g'(y)$ ∎

Cor If $h(x) = \dfrac{1}{f(x)}$ then $h'(x) = -\dfrac{f'(x)}{f(x)^2}$

Pf Say $g(x) = \frac{1}{x}$, then $h(x) = g(f(x))$, so $h'(x) = g'(f(x))\, f'(x)$.

$$= -\frac{1}{f(x)^2} f'(x).$$ ∎

Ex 1) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is discontinuous at $x = 0$ [take e.g. $\varepsilon = \frac{1}{2}$]

2) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$, but not differentiable

[since $\lim_{t\to 0} \dfrac{t\sin\left(\frac{1}{t}\right)}{t} = \sin\left(\frac{1}{t}\right)$ which has no $t \to 0$ limit]

3) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at $x=0$, $f'(0)=0$

but, $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ for $x \neq 0$

so f' is not continuous at $x=0$

Def $f: X \to \mathbb{R}$ has a local maximum at $p \in X$ if $\exists \delta > 0$ s.t.

$d(p,q) < \delta \Rightarrow f(q) \leq f(p)$.

(Similarly define local minimum.)

Thm $f: [a,b] \to \mathbb{R}$, if f has a local maximum at $x \in (a,b)$ and if f is differentiable at x, then $f'(x) = 0$.

Pf Fix δ s.t. $a < x-\delta < x < x+\delta < b$ and $t \in (x-\delta, x+\delta) \Rightarrow f(t) \leq f(x)$.

Then for $t \in (x-\delta, x)$ we have $\dfrac{f(t)-f(x)}{t-x} \leq 0$, so $f'(x) \leq 0$. (why?)

$t \in (x, x+\delta)$ we have $\dfrac{f(t)-f(x)}{t-x} \geq 0$, so $f'(x) \geq 0$.

Thus $f'(x) = 0$. ∎

Thm If $f, g: [a,b] \to \mathbb{R}$ cts on $[a,b]$ diff'ble on (a,b), then $\exists x \in (a,b)$ s.t.

$$(f(b)-f(a))\, g'(x) = (g(b)-g(a))\, f'(x)$$ ("generalized Mean Value Thm")

Pf Let $h(t) = (f(b)-f(a))\, g(t) - (g(b)-g(a))\, f(t)$

Then $h(a) = h(b)$, h cts on $[a,b]$, h diff'ble on (a,b)

Need to show $h'(x) = 0$ for some $x \in (a,b)$.

If h constant, ✓

If $h(y) > 0$ for some $y \in (a,b)$ then let x be the point at which h achieves its maximum. $x \in (a,b)$. Thus $h'(x) = 0$. ✓

If $h(y) < 0$ for some $y \in (a,b)$ then let x be the point at which h achieves its minimum. $x \in (a,b)$. Thus $h'(x) = 0$. ✓ ∎

Cor If $f:[a,b]\to\mathbb{R}$ cts on $[a,b]$ and diff'ble on (a,b) then $\exists\, x\in(a,b)$ s.t.
$$f(b)-f(a)=(b-a)f'(x)$$

Pf Set $g(x)=x$ in the above.

Cor If $f:[a,b]\to\mathbb{R}$ cts on $[a,b]$ and diff'ble on (a,b),

a) if $f'(x)\geq 0$ $\forall x\in(a,b)$ then f is monotonically increasing.

b) if $f'(x)=0$ $\forall x\in(a,b)$ then f is constant

c) if $f'(x)\leq 0$ $\forall x\in(a,b)$ then f is monotonically decreasing.

Pf a) If $a<x_1<x_2<b$ then $\exists\, x\in(x_1,x_2)$ s.t. $f(x_2)-f(x_1)=(x_2-x_1)f'(x)\geq 0$
so $f(x_2)\geq f(x_1)$.

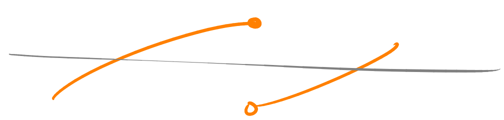
b, c) similar. ∎

Derivatives aren't necessarily continuous, but do satisfy something like Intermediate Value Thm:

Thm $f:[a,b]\to\mathbb{R}$ diff'ble, $f'(a)<\lambda<f'(b)\Rightarrow\exists\, x\in(a,b)$ s.t. $f'(x)=\lambda$.

Pf Define $g:[a,b]\to\mathbb{R}$ by $g(t)=f(t)-\lambda t$.
Then $g'(a)<0$ so $\exists\, t_1\in(a,b)$ s.t. $g(t_1)<g(a)$. (using def. of derivative)
and $g'(b)>0$ so $\exists\, t_2\in(a,b)$ s.t. $g(t_2)<g(b)$.
Thus, if we let x be the point where g attains its minimum, $x\in(a,b)$.
Then $g'(x)=0$, i.e. $f'(x)=\lambda$. ∎

[So, they can't have jumps like



But, we've seen they can be discontinuous.]

Def $f^{\overbrace{\prime\prime\prime\prime\prime\prime}^{n\text{ times}}} = f^{(n)}$. (The domain of $f^{(n)}$ is the subset of the domain of $f^{(n-1)}$ where $f^{(n-1)}$ is diff'ble.)

Thm Say $f:[a,b]\to\mathbb{R}$, $n\in\mathbb{N}$, $f^{(n-1)}$ cts on $[a,b]$ and diff'ble on (a,b).
Say $\alpha,\beta\in[a,b]$, $\alpha\neq\beta$.
Define
$$P_n(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$
Then $\exists x\in(\alpha,\beta)$ s.t.
$$f(\beta)=P_n(\beta)+\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

Pf Let M be defined by $f(\beta)=P_n(\beta)+M(\beta-\alpha)^n$; want to show $\exists x$ s.t. $M=\frac{f^{(n)}(x)}{n!}$
Let $g(t)=f(t)-P_n(t)-M(t-\alpha)^n$
$$g^{(n)}(t)=f^{(n)}(t)-n!M$$
So, want to show $\exists x\in(\alpha,\beta)$ s.t. $g^{(n)}(x)=0$.
We have $g(\alpha)=g'(\alpha)=\cdots=g^{(n-1)}(\alpha)=0$, and $g(\beta)=0$.
Thus $\exists x_1\in(\alpha,\beta)$ s.t. $g'(x_1)=0$.
Similarly $\exists x_2\in(\alpha,x_1)$ s.t. $g''(x_2)=0$.
$\vdots$
$\exists x_n\in(\alpha,x_{n-1})$ s.t. $g^{(n)}(x_n)=0$. ∎

Rk If we can control $|f^{(n)}(x)|$ somehow, this formula gives a good approximation to $f(\beta)$; sometimes even $P_n(\beta)\to f(\beta)$.
(e.g. if $f(x)=e^x$ or $f(x)=\sin(x)$)
But, warning: <u>even if</u> $\{P_n(x)\}$ converges, it <u>might not</u> converge to $f(x)$!

Ex $f(x)=\begin{cases} e^{-1/x^2} & x>0\\ 0 & x=0\end{cases}$ has $f^{(n)}(0)=0$ $\forall n\in\mathbb{N}$, hence $P_n(\beta)=0$

Thm Suppose $f, g : (a,b) \to \mathbb{R}$ differentiable, $g'(x) \neq 0$ $\forall x \in (a,b)$,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A, \text{ and } \lim_{x\to a} f(x) = 0, \lim_{x \to a} g(x) = 0.$$

Then, $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = A.$

Pf Pick $\varepsilon > 0$. Pick $r \in \mathbb{R}$ s.t. $A < r < A + \varepsilon$.

Since $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, $\exists c \in (a,b)$ s.t. $a < t < c \Rightarrow \dfrac{f'(t)}{g'(t)} < r.$

For any $x, y \in \mathbb{R}$ s.t. $a < x < y < c$, by generalized Mean Value Thm,

$$\exists t \in (x,y) \text{ s.t. } \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Thus, $\displaystyle\lim_{x \to a} \frac{f(x) - f(y)}{g(x) - g(y)} \leq r$, i.e. $\dfrac{f(y)}{g(y)} \leq r$, so $\dfrac{f(y)}{g(y)} < A + \varepsilon.$

Altogether, we've shown that if $\varepsilon > 0$ then $\exists c \in (a,b)$ s.t. $a < y < c \Rightarrow \dfrac{f(y)}{g(y)} < A + \varepsilon.$

Similarly show that if $\varepsilon > 0$ then $\exists c' \in (a,b)$ s.t. $a < y < c' \Rightarrow \dfrac{f(y)}{g(y)} > A - \varepsilon.$

Then it follows that $\displaystyle\lim_{y \to a} \frac{f(y)}{g(y)} = A.$ (Take $\delta = \min(|c-a|, |c'-a|)$ in def. of limit) ∎

Rk There are other variants of this rule involving limits as $x \to \infty$ or limits where $g(x) \to \infty$ as $x \to a$. For these see Rudin.