The Riemann Integral

<u>Def</u> A <u>partition</u> P of $[a,b]$ is a set of points $x_0, \dots, x_n$
with $a = x_0 \le x_1 \le \cdots \le x_n = b$

Let $\Delta x_i = x_i - x_{i-1}$

If $f: [a,b] \to \mathbb{R}$ bounded, set:

$$M_i = \sup \{f(x) \mid x_{i-1} \le x \le x_i\}$$
$$m_i = \inf \{f(x) \mid x_{i-1} \le x \le x_i\}$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\overline{\int_a^b} f(x)\,dx = \inf \{U(P,f) \mid P \text{ a partition of } [a,b]\}$$ ("upper Riemann integral")

$$\underline{\int_a^b} f(x)\,dx = \sup \{L(P,f) \mid P \text{ a partition of } [a,b]\}$$ ("lower Riemann integral")

<u>Def</u> If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, say f is <u>Riemann integrable</u> on $[a,b]$

and let $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

<u>Def</u> Partition P^* is a <u>refinement</u> of P if every point of P is also a point of P^*.
Given P_1, P_2 their <u>common refinement</u> is the partition consisting of all points of P_1 and P_2.

<u>Prop</u> If P^* is a refinement of P, then $L(P,f,\alpha) \le L(P^*,f,\alpha)$
$U(P,f,\alpha) \ge U(P^*,f,\alpha)$

<u>Pf</u> It's enough to deal with the case where P^* contains 1 more pt than P: $x_{i-1} < x^* < x_i$
Say $w_1 = \inf\{f(x) \mid x_{i-1} \le x \le x^*\}$, $w_2 = \inf\{f(x) \mid x^* \le x \le x_i\}$
Then $w_1 \ge m_i$, $w_2 \ge m_i$,

$$L(P^*,f) - L(P,f) = [\omega_1(x^* - x_{i-1}) + \omega_2(x_i - x^*)] - m_i(x_i - x_{i-1})$$
$$= (\omega_1 - m_i)(x^* - x_{i-1}) + (\omega_2 - m_i)(x_i - x^*)$$
$$\geq 0$$

Similar for U. ∎

Prop $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

Pf Just need to show $L(P_1,f) \leq U(P_2,f)$ for all partitions P_1, P_2.

Let P^* be common ref of P_1, P_2.

Then $L(P_1,f) \leq L(P^*,f) \leq U(P^*,f) \leq U(P_2,f)$ ∎

Prop f is Riemann integrable $\iff$ $\forall \varepsilon > 0$, $\exists$ partition P s.t.
$$U(P,f) - L(P,f) < \varepsilon.$$

Pf $(\Leftarrow)$ $\forall \varepsilon>0$, $0 \leq \overline{\int} f(x)\,dx - \underline{\int} f(x)\,dx < \varepsilon \implies \overline{\int} f(x)\,dx = \underline{\int} f(x)\,dx$

$(\Rightarrow)$ $\forall \varepsilon>0$, $\exists P_1$ s.t. $\int f(x)\,dx - L(P_1,f) < \varepsilon/2$

$\exists P_2$ s.t. $U(P_2,f) - \int f(x)\,dx < \varepsilon/2$

Let P be the common ref. of P_1, P_2

then $U(P,f) \leq U(P_2,f) < \int f(x)\,dx + \varepsilon/2 < L(P_1,f) + \varepsilon \leq L(P,f) + \varepsilon$ ∎

Prop If $U(P,f) - L(P,f) < \varepsilon$

a) if P^* is a refinement of P then $U(P^*,f) - L(P^*,f) < \varepsilon$.

b) if $s_i, t_i \in [x_{i-1}, x_i]$ then

b1) $\sum_{i=1}^{n} |f(s_i) - f(t_i)|\,\Delta x_i < \varepsilon$.

b2) if f is Riemann integrable then $\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)\,dx\right| < \varepsilon$

(so, can use any sampling points.)

Pf a) use $U(P^*,f) < U(P,f)$ and $L(P^*,f) > L(P,f)$.

b1) $\sum_{i=1}^{n} |f(s_i)-f(t_i)| \Delta x_i < \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = U(P,f) - L(P,f)$.

b2) use $L(P,f) \leq \sum f(t_i) \Delta x_i \leq U(P,f)$

$L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f)$

Thm $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow$ f is Riemann integrable on $[a,b]$.

Pf Say $\varepsilon > 0$. Pick $\eta < \frac{\varepsilon}{b-a}$.

f is cts on $[a,b] \Rightarrow f$ is uniformly cts on $[a,b]$.

Thus $\exists \delta > 0$ s.t. $|x-t| < \delta \Rightarrow |f(x)-f(t)| < \eta$.

Now let P be any partition of $[a,b]$ s.t. $\Delta x_i < \delta$ for all i. Then $M_i - m_i \leq \eta$,

so $U(P,f) - L(P,f) < \eta \cdot (b-a) = \varepsilon$. ∎

Rk 1) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$ is not Riemann integrable on any $[a,b]$

2) Changing f at finite # pts doesn't affect $\int f(x)\,dx$.

Thm $f: [a,b] \to \mathbb{R}$ monotonic $\Rightarrow$ f is Riemann integrable on $[a,b]$.

Pf Say $\varepsilon > 0$. Fix $n \in \mathbb{N}$. Pick P s.t. $\Delta x_i = \frac{b-a}{n}$

Suppose f monotonic increasing.

Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$.

$$U(P,f) - L(P,f) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{b-a}{n}(f(b) - f(a)) < \varepsilon \qquad \text{if } n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$$ ∎

Thm Suppose f is bounded on $[a,b]$ and f is discontinuous only at finitely many points of $[a,b]$. Then f is Riemann integrable on $[a,b]$.

Pf Let $M = \sup \{|f(t)| \mid t \in [a,b]\}$.

Say $\varepsilon > 0$. Take intervals (u_j, v_j) containing the discontinuities of f, with $\sum_{j=1}^{n} v_j - u_j < \frac{\varepsilon}{4M}$. Then f is unif cts on $K = [a,b] \setminus \{(u_1,v_1) \cup \cdots \cup (u_n,v_n)\}$.

$s,t \in K$: $|s-t| < \delta \Rightarrow |f(s)-f(t)| < \frac{\varepsilon}{2(b-a)}$.

Now take a partition P including all u_j, v_j, and s.t. $\Delta x_i < \delta$ except for $x_{i-1} \in \{u_1, \ldots, u_n\}$.

$$U(P,f) - L(P,f) \leq 2M \cdot \left(\frac{\varepsilon}{4M}\right) + (b-a) \cdot \frac{\varepsilon}{2(b-a)} < \varepsilon. \quad \blacksquare$$

Thm Suppose f is Riemann integrable on $[a,b]$, $f: [a,b] \to [m,M] \subset \mathbb{R}$, $\phi: [m,M] \to \mathbb{R}$ cts. Let $h: [a,b] \to \mathbb{R}$ be $h(x) = \phi(f(x))$. Then h is Riemann integrable on $[a,b]$.

Pf Fix $\varepsilon > 0$. Let $K = \sup\{|\phi(t)| \mid m \leq t \leq M\}$, $\varepsilon' = \dfrac{\varepsilon}{b-a+2K}$.

ϕ is unif cts on $[m,M]$. So, $\exists\, \delta > 0$ s.t. $\delta < \varepsilon'$ and $\underset{s,t \in [m,M]}{|s-t| \leq \delta} \Rightarrow |\phi(s) - \phi(t)| \leq \varepsilon'$.

Pick a partition P of $[a,b]$ s.t. $U(P,f) - L(P,f) < \delta^2$.

Let:

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$
$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

$$M_i^* = \sup\{h(x) \mid x_{i-1} \leq x \leq x_i\}$$
$$m_i^* = \inf\{h(x) \mid x_{i-1} \leq x \leq x_i\}$$

$$\{1, \ldots, n\} = A \cup B \qquad \begin{array}{l} A = \{i \mid M_i - m_i < \delta\} \\ B = \{i \mid M_i - m_i \geq \delta\} \end{array}$$

For $i \in A$, $M_i^* - m_i^* \leq \varepsilon'$ (since $M_i^* = \phi(s_i)$ and $m_i^* = \phi(t_i)$, with $|s_i - t_i| < \delta$)

For $i \in B$, $M_i^* - m_i^* \leq 2K$.

$$\delta \cdot \sum_{i \in B}' \Delta x_i \leq \sum_{i \in B}' (M_i - m_i)\Delta x_i < U(P^*,f) - L(P^*,f) < \delta^2$$

so $\displaystyle\sum_{i \in B}' \Delta x_i < \delta$.

$$\text{Thus } U(P,f) - L(P,f) \leq \underbrace{\varepsilon'(b-a)}_{\text{from } A} + \underbrace{2K\delta}_{\text{from } B} < \varepsilon'(b-a+2K) = \varepsilon. \quad \blacksquare$$

Prop a) f_1, f_2 Riem. integrable on $[a,b] \Rightarrow f_1+f_2$ Riem integrable on $[a,b]$

$$\int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx = \int_a^b (f_1+f_2)(x)\,dx$$

$c \in \mathbb{R} \Rightarrow cf$ Riem integrable on $[a,b]$ $\quad \int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$

b) if $f_1(x) \leq f_2(x)$ $\forall x \in [a,b]$ and f_1, f_2 Riem integrable on $[a,b]$, then

$$\int_a^b f_1(x)\,dx \leq \int_a^b f_2(x)\,dx$$

c) if $f(x)$ Riem integrable on $[a,b]$ and $a<c<b$ then $f(x)$ is Riem integrable on $[a,c]$ and $[c,b]$, and

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

d) if $f(x)$ Rieman integrable on $[a,b]$ and $|f(x)| \leq M$ $\forall x \in [a,b]$ then

$$\left|\int_a^b f(x)\,dx\right| \leq M|b-a|$$

Pf of c) Pick a partition P of $[a,b]$ such that $U(P,f)-L(P,f) < \varepsilon$.

Let P^* be refinement of P to include c. $U(P^*,f)-L(P^*,f)<\varepsilon$ also.

Divide P^* into partitions P_1 and P_2 of $[a,c]$ and $[c,b)$ respectively.

Then $U(P_1,f)-L(P_1,f) < U(P,f)-L(P,f) < \varepsilon$

$U(P_2,f)-L(P_2,f) < U(P,f)-L(P,f) < \varepsilon$

Thus f is indeed integrable on $[a,b]$ and $[b,c]$.

Also, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \geq L(P_1,f)+L(P_2,f) = L(P^*,f) > \int_a^b f(x)\,dx - \varepsilon$

Thus $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \geq \int_a^b f(x)\,dx$

and similarly prove $\leq$ ∎

Thm If f, g both Riem integrable on $[a,b]$ then fg is Riem integrable on $[a,b]$.

Pf We showed f integrable $\Rightarrow f^2$ integrable (since $\phi(x) = x^2$ is cts)

And $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$. $\blacksquare$

Thm If f Riem integrable on $[a,b]$ then $|f|$ Riem integrable on $[a,b]$;
$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Pf $\phi(x) = |x|$ is cts, so $|f|$ is integrable.

$$\begin{aligned}\left|\int f(x)\,dx\right| &= c\int f(x)\,dx \qquad (c = \pm 1)\\ &= \int cf(x)\,dx \\ &\leq \int |f(x)|\,dx\end{aligned}$$ $\blacksquare$

Thm If: $f : [a,b] \to \mathbb{R}$ Rieman integrable on $[a,b]$
$\varphi : [A,B] \to [a,b]$ strictly increasing, differentiable
φ' Rieman integrable on $[A,B]$ "$x = \varphi(y)$"
$g : [A,B] \to \mathbb{R}$ defined by $g(y) = f(\varphi(y))$

Then: $g(y)\varphi'(y)$ is Rien integrable on $[A,B]$ and $\int_a^b f(x)\,dx = \int_A^B g(y)\varphi'(y)\,dy$

Pf Fix $\varepsilon > 0$.

Let P be a partition of $[A,B]$ s.t. $U(P,\varphi') - L(P,\varphi') < \varepsilon$.

$A = y_0 \leq \cdots \leq y_n = B$

Let $\tilde{P}$ be the partition of $[a,b]$ by $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ with $x_i = \varphi(y_i)$.

$\exists\, t_i \in [y_{i-1}, y_i]$ s.t. $x_i - x_{i-1} = (y_i - y_{i-1})\varphi'(t_i)$.

Let $M = \sup\{|f(x)| \mid a \leq x \leq b\}$.

For any $s_i \in [y_{i-1}, y_i]$ we have

$\sum_{i=1}^n |\varphi'(s_i) - \varphi'(t_i)| < \varepsilon$, so $\sum_{i=1}^n |f(s_i)\varphi'(s_i) - f(s_i)\varphi'(t_i)| < M\varepsilon$

and thus

$$\left|\sum_{i=1}^n f(s_i)(y_i - y_{i-1})\varphi'(s_i) - \sum_{i=1}^n f(s_i)(x_i - x_{i-1})\right| < M\varepsilon$$

Thus $\sum_{i=1}^n f(s_i)(x_i - x_{i-1}) \leq \sum f(s_i)(y_i - y_{i-1})\varphi'(s_i) + M\varepsilon \leq U(P, f\varphi') + M\varepsilon$ for all choices of s_i

and hence $U(\tilde{P}, f) \le U(P, f\varphi') + M\varepsilon$

similarly $U(P, f\varphi') \le U(\tilde{P}, f) + M\varepsilon$

so $|U(P, f\varphi') - U(\tilde{P}, f)| < M\varepsilon$

and the same holds for any <u>refinement</u> of P, thus

$$\left| \overline{\int_a^b} f(x)\,dx - \overline{\int_A^B} g(y)\,\varphi'(y)\,dy \right| < M\varepsilon$$

and this holds for any ε so finally we get

$$\overline{\int_a^b} f(x)\,dx = \overline{\int_A^B} g(y)\,\varphi'(y)\,dy$$

Similarly for $\underline{\int}$. ∎

<u>Thm</u> Say f Riemann int'ble on $[a,b]$. For $a \le x \le b$ set $F(x) = \int_a^x f(t)\,dt$.

Then: 1) F is cts on $[a,b]$

2) If f cts at $x_0 \in [a,b]$ then F is diff'ble at x_0, $F'(x_0) = f(x_0)$.

<u>Pf</u> f is bounded; say $|f(t)| \le M$ $\forall t \in [a,b]$.

If $a \le x < y \le b$ then $|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \le M(y-x)$.

Thus, if $\varepsilon > 0$, take $\delta = \varepsilon/M$; if $|x-y| < \delta$ then $|F(x) - F(y)| < \varepsilon$.

So F is cts.

Say f is cts at x_0. Given $\varepsilon > 0$ fix $\delta > 0$ s.t. $|x_0 - t| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon$

Thus if $x_0 - \delta < s \le x_0 \le t < x_0 + \delta$ and $a \le s < t \le b$

then $$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)]\,du \right| < \varepsilon$$

Pick $s = x_0$, then $\lim_{t \to x_0} \dfrac{F(t) - F(x_0)}{t - x_0} - f(x_0) = 0$

ie $F'(x_0) - f(x_0) = 0$ ∎

Cor Say f cts on $[a,b]$. Suppose $F:[a,b]\to\mathbb{R}$ diff'ble with $F'=f$.
Then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Pf By the Thm above, $F(x)$ and $\int_a^x f(x)\,dx$ have same derivative.
Thus $\exists C\in\mathbb{R}$ s.t. $F(x)=\int_a^x f(x)\,dx + C$.
Then $F(b)-F(a)=\int_a^b f(x)\,dx$ as desired. $\blacksquare$

Rk Rudin proves a slightly stronger version, where f is only integrable, not nec. cts.

Cor Say $F,G:[a,b]\to\mathbb{R}$ diff'ble, $F'=f$, $G'=g$ both cts on $[a,b]$.
Then $\int_a^b F(x)g(x)\,dx = F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)\,dx$.

Pf Apply the last result to $H(x)=F(x)G(x)$.