Uniform convergence

Some cautionary tales:

1) Recall: if $f_n(x) = x^n$

then $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

So, even though each $f_n(x)$ is cts on $[0,1]$

the limit $f(x) = \lim_{n\to\infty} f_n(x)$ is <u>not</u> cts.

3) If $a_{m,n} = \begin{cases} 1 & m \geq n \\ 0 & m < n \end{cases}$

then $\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = 0$

$\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} = 1$

2) If $f_n(x) = n^2 x (1-x^2)^n$

then $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1]$

so $\int_0^1 \lim_{n\to\infty} f_n(x)\, dx = 0$

But $\int_0^1 f_n(x) = n^2 \int_0^1 x(1-x^2)^n\, dx = \frac{n^2}{2}\int_0^1 (1-u)^n\, du = \frac{n^2}{2(n+1)}$

so $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \lim_{n\to\infty} \frac{n^2}{2n+2} = \infty$

Thus, interchanging limits and integrals isn't always OK!

Def Given a seq. $\{f_n\}$ of functions $f_n : X \to Y$, and $f: E \to Y$, $E \subset X$

1) $\{f_n\}$ <u>converges pointwise to f on E</u> if $\forall x \in E$, $f_n(x) \to f(x)$
(i.e. $\forall x \in X$, $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon$)

2) $\{f_n\}$ <u>converges uniformly to f on E</u> if, $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall x \in E$, $n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon$

Def Given a seq. $\{f_n\}$ of functions $f_n : X \to \mathbb{R}$, and $s: X \to \mathbb{R}$,
$\sum_{n=1}^{\infty} f_n(x)$ <u>converges uniformly</u> to $s(x)$ if the partial sums $s_n(x)$ converge uniformly to $s(x)$.

Prop $\{f_n\}$ converges uniformly on $E \iff \forall \varepsilon>0\ \exists N$ s.t. $m,n \geq N,\ x\in E \Rightarrow$
$|f_n(x)-f_m(x)|<\varepsilon$

Pf $(\Rightarrow)$ Say $f_n \to f$ uniformly on E. Fix $\varepsilon>0$.
Then $\exists N$ s.t. $n \geq N,\ x\in E \Rightarrow |f_n(x)-f(x)| < \varepsilon/2$.
Thus $n,m \geq N,\ x \in E \Rightarrow$

$$|f_n(x)-f_m(x)| \leq |f_n(x)-f(x)| + |f(x)-f_m(x)| < \varepsilon.$$

$(\Leftarrow)$ $\{f_n(x)\}$ is Cauchy for each $x\in E$. So, $\{f_n\}$ converges pointwise to some f. Fix $\varepsilon>0$ and fix N s.t. $m,n\geq N,\ x\in E \Rightarrow |f_n(x)-f_m(x)|<\frac{\varepsilon}{2}$.
Then taking $m\to\infty$ gives $|f_n(x)-f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$. ∎

Prop For $f_n: X\to\mathbb{R}$, let $M_n = \sup\{|f_n(x)-f(x)| : x\in E\}$
Then $f_n\to f$ uniformly on $E \iff \lim_{n\to\infty} M_n = 0$.

Pf Exercise.

Prop Say $f_n: X\to\mathbb{R}$, $|f_n(x)| \leq M_n\ \ \forall x\in E$, and $\sum_{n=1}^{\infty} M_n$ converges.
Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

Pf Fix $\varepsilon>0$. Then $\exists N$ s.t. $n,m\geq N \Rightarrow \sum_{i=n}^{m} M_i < \varepsilon$

Then, $\forall x\in E\ \ n,m\geq N \Rightarrow \left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} M_i < \varepsilon$

ie $|s_n(x)-s_m(x)| < \varepsilon$ ∎

Ex $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$.

Thm If: $f_n: X\to\mathbb{R}$ $\quad E\subset X$
$f_n\to f$ uniformly on E, x limit pt of E, $\lim_{t\to x} f_n(t) = A_n$

Then: $\{A_n\}$ converges, $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

(ie, $\lim_{t\to x}\lim_{n\to\infty} f_n = \lim_{n\to\infty}\lim_{t\to x} f_n$)

Pf Fix $\varepsilon>0$.

$\exists N$ s.t. $n,m\geq N,\ t\in E \Rightarrow |f_n(t)-f_m(t)|<\varepsilon/2$

Take limit $t\to x$: $|A_n-A_m|\leq \varepsilon/2<\varepsilon$

Thus $\{A_n\}$ is Cauchy $\Rightarrow$ converges; call the limit A.

$$|f(t)-A|\leq|f(t)-f_n(t)|+|f_n(t)-A_n|+|A_n-A|$$

Fix N_1 s.t. $|f(t)-f_n(t)|<\varepsilon/3\ \ \forall t\in E,\ n\geq N_1$

N_2 s.t. $|A_n-A|<\varepsilon/3\ \ \forall n\geq N_2$

$n=\max(N_1,N_2)$

Fix a nbhd V of x s.t. $t\in V\cap E,\ t\neq x \Rightarrow |f_n(t)-A_n|<\varepsilon/3$

Then $|f(t)-A|<\varepsilon\ \ \forall t\in V\cap E$.

Cor If: $f_n: X\to\mathbb{R}$ $\quad E\subset X$

$f_n\to f$ uniformly on E

f_n cts on E

Then: f is cts on E

A partial converse:

Thm If: K compact,

$\{f_n\}$ sequence of cts functions $f_n: K\to\mathbb{R}$,

$f_n\to f$ pointwise with $f: K\to\mathbb{R}$ cts

$f_n(x)\geq f_{n+1}(x)\ \ \forall x\in K,\ n\in\mathbb{N}$,

Then: $f_n\to f$ uniformly on K.

Pf Say $g_n=f_n-f$. g_n is cts, $g_n\to 0$ pointwise, $g_n\geq g_{n+1}$. Want: $g_n\to 0$ uniformly.

Say $\varepsilon>0$. Let $K_n=\{x\in K\mid g_n(x)\geq\varepsilon\}$. $K_n\supset K_{n+1}$, all K_n compact.

But $\bigcap_{n=1}^{\infty}K_n=\phi$. (Why?) Thus, $K_N=\phi$ for some N. So for $n\geq N$ we have $g_n(x)<\varepsilon$. Thus $g_n\to 0$ uniformly as desired.

Def X metric space: $\mathcal{C}(X) = \{\text{continuous bounded functions } f: X \to \mathbb{R}\}$

Def For $f \in \mathcal{C}(X)$, $\|f\| = \sup\{|f(x)| \mid x \in X\}$

Prop Defining $d(f,g) = \|f-g\|$ makes $\mathcal{C}(X)$ into a metric space.

Pf For any $f,g \in \mathcal{C}(X)$,

$\|f-g\| = 0 \Rightarrow f(x)-g(x) = 0 \ \forall x \in X \Rightarrow f = g.$ ✓

Also, $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|+\|g\| \quad \forall x \in X$

thus $\|f+g\| \leq \|f\|+\|g\|$

Thus for any $F,G,H \in \mathcal{C}(X)$,

set $\begin{array}{l} f = G-H \\ g = H-F \end{array}$ to get $\|G-F\| \leq \|G-H\|+\|H-F\|$ ✓

Prop $\mathcal{C}(X)$ is complete.

Pf Say $\{f_n\}$ Cauchy in $\mathcal{C}(X)$.

Then $\forall \varepsilon > 0 \ \exists N$ s.t. $n,m \geq N \Longrightarrow \|f_m(x) - f_n(x)\| < \varepsilon \ \forall x \in X$.

We've proved this implies $\{f_n\}$ converges uniformly to some f, and that this f is cts. Also, $\exists N$ s.t. $\|f_N(x) - f(x)\| < 1 \ \forall x \in X$, and f_N bounded, so f bounded. Thus $f \in \mathcal{C}(X)$. ∎

Thm Say f_n Riem. int'ble on $[a,b]$ for $n \in \mathbb{N}$, $f_n \to f$ uniformly on $[a,b]$. Then f Riem int'ble on $[a,b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

Pf Set $\varepsilon_n = \sup\{|f_n(x) - f(x)| : x \in [a,b]\} = \|f_n - f\|$

Then $\quad f_n(x) - \varepsilon_n \leq f(x) \leq f_n(x) + \varepsilon_n \ \forall x \in [a,b]$

So $\int_a^b (f_n(x) - \varepsilon_n)\,dx \leq \underline{\int} f(x)\,dx \leq \overline{\int} f(x)\,dx \leq \int_a^b (f_n(x) + \varepsilon_n)\,dx$

Thus $0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq 2\varepsilon_n (b-a)$

And $\lim_{n\to\infty} \varepsilon_n = 0$, so $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ ie f is integrable on $[a,b]$

also $\left| \int_a^b f_n(x) - \int_a^b f(x)\,dx \right| < \varepsilon_n$

so $\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx$ ∎

Cor If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converging uniformly on $[a,b]$

then $\int_a^b f(x)\,dx = \sum \int_a^b f_n(x)\,dx$

Thm Suppose $\{f_n\}$ seq. of f's diff'ble on $[a,b]$, $x_0 \in [a,b]$ and $\{f_n(x_0)\}$ converges.
Suppose $\{f_n'\}$ converges uniformly on $[a,b]$.
Then $\{f_n\}$ converges uniformly on $[a,b]$ to some f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$

Ex Defining $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

These converge uniformly on any $[a,b]$ (not on the whole $\mathbb{R}$ though)
and we have $\sin'(x) = \cos(x)$

Pf Fix $\varepsilon > 0$. $\exists N$ s.t. $n,m \geq N \Rightarrow |f_n(x_0) - f_m(x_0)| < \varepsilon/2$,

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \quad \forall t \in [a,b]$$

Using mean value thm on $f_n - f_m$, $\quad |f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|\,\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2}$
for all $x, t \in [a,b]$.

Thus $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n,m \geq N$ and $x \in [a,b]$

So $\{f_n\}$ converges uniformly on $[a,b]$, to some f.

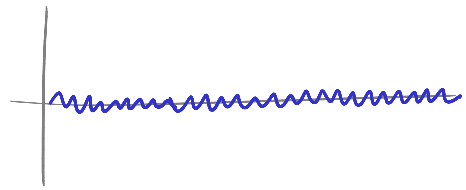
Let $\phi_n(t) = \dfrac{f_n(t) - f_n(x)}{t - x}$ $\quad \phi(t) = \dfrac{f(t) - f(x)}{t-x}$. $\quad \phi_n \to \phi$.

But $|\phi_n(t) - \phi_m(t)| \leq \dfrac{\varepsilon}{2(b-a)}$, so $\phi_n \to \phi$ uniformly

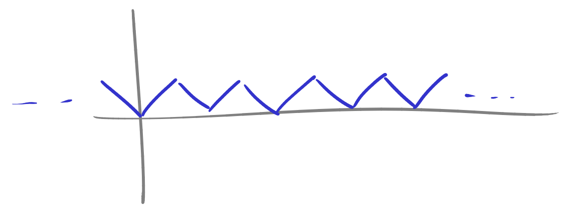
Also $\lim_{t \to x} \phi_n(t) = f_n'(x)$

Thus $\lim_{n\to\infty} f_n'(x) = \lim_{t \to x} \phi(t) = f'(x)$

Rk Uniform convergence of $\{f_n\}$ is not enough!
$\{f_n\}$ could be getting very "spiky".



Ex Define $\varphi(x) = |x|$ if $-1 \leq x \leq 1$
and $\varphi(x+2) = \varphi(x)$



φ is cts, $|\varphi(s) - \varphi(t)| \leq |s - t|$.

Define $f_n(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$

$\{f_n\}$ converges uniformly to a cts $f(x)$. (since $\sum \left(\frac{3}{4}\right)^n$ converges.)

Now fix x, m; set $\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$ so $\nexists$ integer between $4^m x$, $4^m(x+\delta_m)$

and set $\gamma_n = \dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$

For $n > m$, $\gamma_n = 0$.

For $0 \leq n \leq m$, $|\gamma_n| \leq 4^n$.

and $|\gamma_m| = 4^m$.

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1)$$

But as $m \to \infty$, $\delta_m \to 0$.

Thus $f(x)$ is not differentiable at x for any x!