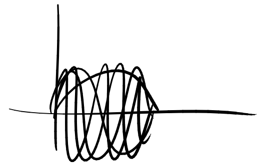
Does a seq. of bdd functions have a uniformly convergent subsequence?

No. 1) $f_n(x) = \sin(nx)$



2) $f_n(x) = \dfrac{x^2}{x^2+(1-nx)^2}$ uniformly bdd on $[0,1]$, $f_n \to 0$ pointwise,

but $f_n(\frac{1}{n}) = 1$ so no subseq. can have $f_{n_k} \to 0$

To improve things:

Def A family $\mathcal{F}$ of functions $f: E \to \mathbb{R}$ is equicontinuous (or uniformly equicontinuous) if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d(x,y) < \delta,\ f \in \mathcal{F} \Rightarrow |f(x) - f(y)| < \varepsilon$

Prop $\{f_n\}$ pointwise bdd sequence of functions $E \to \mathbb{R}$, E countable

$\Rightarrow \exists$ subseq $\{f_{n_k}\}$ s.t. $\{f_{n_k}(x)\}$ converges $\forall x \in E$.

Pf $E = \{x_1, x_2, \ldots\}$

Make sequences

$S_1 = f_{1,1} \quad f_{1,2} \quad \cdots$

$S_2 = f_{2,1} \quad f_{2,2} \quad \cdots$

$S_3 = f_{3,1} \quad f_{3,2} \quad \cdots$

$\vdots$

where S_n is subseq of S_{n-1} $\forall n \geq 2$

$\{f_{n,k}(x_n)\}$ converges

(use boundedness of $\{f_m(x_n)\}$)

Then take the seq $\{f_{n,n}\}$ ∎

Thm (Arzela-Ascoli)

K compact, $f_n \in \mathcal{C}(K)$ $\forall n \geq 1$, $\{f_n\}$ pointwise bdd equicontinuous

$\Rightarrow$ 1) $\{f_n\}$ uniformly bdd

2) $\{f_n\}$ has uniformly convergent subsequence

Pf 1) Say $\varepsilon > 0$. Pick $\delta > 0$ s.t. $d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \ \forall n$

K compact $\Rightarrow \exists \{p_1, \ldots, p_n\}$ s.t. the $N_\delta(p_i)$ cover K.

$\{f_n\}$ pointwise bdd $\Rightarrow \forall\ 1 \le i \le n,\ \exists M_i$ s.t. $|f_n(p_i)| < M_i\ \forall n$

then $\max(M_i) + \varepsilon$ gives uniform bound for f_n

2) By a HW problem, $\exists\ E \subset K$ countable dense.

We know $\exists$ a subseq. $\{f_{n_i}\}$ converging on E.

Let $g_i = f_{n_i}$. We'll show $\{g_i\}$ converges uniformly on K.

Say $\varepsilon > 0$. Pick $\delta > 0$ s.t. $d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}\ \forall n$

(as before)

E dense, K compact $\Rightarrow \exists\ \{x_1, \ldots, x_m\}$ in $\underline{E}$ s.t. the $N_\delta(x_s)$ cover K.

$\{g_i(x)\}$ converge on $E \Rightarrow \exists N$ s.t. $|g_i(x_s) - g_j(x_s)| < \frac{\varepsilon}{3}\ \forall i,j \ge N,\ 1 \le s \le m.$

For $x \in K$, $x \in N_\delta(x_s)$ for some s.

Thus $\forall i,j \ge N$, $|g_i(x) - g_j(x)| \le |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$

$\le \varepsilon$

$\blacksquare$

Application: solvability of 1^{st}-order initial value prob! (See HW)