Lecture 5

X metric space

Def 1) $A \subset X$, $B \subset X$ are <u>separated</u> if $A \cap \bar{B} = \phi$, $B \cap \bar{A} = \phi$.

2) $E \subset X$ is <u>connected</u> if $\nexists A, B \subset X$ s.t. A, B are separated nonempty and $E = A \cup B$.

Thm $E \subset \mathbb{R}$ connected $\Longleftrightarrow$ if $x \in E$, $y \in E$, $x < z < y$, then $z \in E$.

Pf $(\Rightarrow)$ Say $x \in E$, $y \in E$ but $\exists z$ with $x < z < y$, $y \notin E$.

Let $A = \{w \in E \mid w < z\}$, $B = \{w \in E \mid w > z\}$.

A, B are separated, and $A \cup B = E$. So E is not connected.

$(\Leftarrow)$ Say E is not connected. Then $E = A \cup B$ with A, B separated.

Now pick $x \in A$, $y \in B$, assume $\underline{x < y}$ (can arrange this by swapping A, B if needed)

Set $z = \sup(A \cap [x,y])$. $z \in \overline{(A \cap [x,y])} \subset \bar{A}$, so $z \notin B$, so $x \leq z < y$.

If $z \notin A$ then $x < z < y$ and $z \notin E$.

If $z \in A$ then $z \notin \bar{B}$, so $\exists z_1$ s.t. $z < z_1 < y$ and $z_1 \notin B$. Thus $x < z_1 < y$, $z_1 \notin E$.