Series

Def For a sequence $\{a_n\}$ in $\mathbb{R}$, $\sum_{n=p}^{q} a_n$ means $a_p + a_{p+1} + \cdots + a_q$.

The partial sums of $\{a_n\}$ are $s_n = \sum_{k=1}^{n} a_k$.

If $\exists s \in \mathbb{R}$ s.t. $\{s_n\} \to s$ we write $\sum_{k=1}^{\infty} a_k = s$, and say the sum converges; otherwise the sum diverges.

Thm $\sum a_n$ converges $\iff$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$

Pf This is just the condition that $\{s_n\}$ is Cauchy

Cor $\sum a_n$ converges $\Rightarrow$ $a_n \to 0$.

Thm If $a_n \geq 0$ $\forall n$, then $\sum a_n$ converges $\iff$ partial sums of $\sum a_n$ are a bounded sequence.

Pf Partial sums are monotone.

Thm 1) If $\exists N_0$ s.t. $\forall n \geq N_0$ $|a_n| \leq c_n$ and $\sum c_n$ converges then $\sum a_n$ converges.

2) If $a_n \geq d_n \geq 0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Pf 1) $\exists N \geq N_0$ s.t. $\forall m \geq n \geq N$,

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \varepsilon$$

2) follows from 1). ∎

Thm 1) If $0 \leq x < 1$, then $\sum x^n = \frac{1}{1-x}$

2) $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$

Pf See Rudin

Rk: #1 uses $\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}$

#2 comes down to checking boundedness

e.g. for $p > 1$, use $\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n^p} \leq 2^k \cdot \frac{1}{2^{kp}} = \frac{1}{2^{k(p-1)}}$

Thm (Root Test) Let $\alpha = \limsup \sqrt[n]{|a_n|}$

Then $\sum a_n$ $\begin{cases} \text{converges if } \alpha < 1 \\ \text{diverges if } \alpha > 1 \end{cases}$

Pf If $\alpha < 1$, take β s.t. $\alpha < \beta < 1$ and N s.t. $\forall n \geq N$, $\sqrt[n]{|a_n|} < \beta$ (why does this exist?)
Then for $n \geq N$, $|a_n| < \beta^n$. Then use comparison
If $\alpha > 1$, take γ with $\alpha > \gamma > 1$.
$\exists$ a subseq with $\sqrt[k_n]{|a_{k_n}|} \to \gamma$. Thus $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[k_n]{|a_{k_n}|} > 1$, ie $|a_{k_n}| > 1$.
but then $\{a_n\}$ cannot be converging to 0. ∎

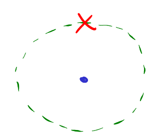
Def Given a sequence $\{c_n\}$, and $x \in \mathbb{R}$, the series $\sum c_n x^n$ is called a <u>power series</u>.

Thm Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$, $x \in \mathbb{R}$.

Then $\sum c_n x^n$ <u>converges</u> if $|x| < R$, <u>diverges</u> if $|x| > R$.

Pf Root test: $\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = \limsup_{n\to\infty} |x| \sqrt[n]{|c_n|} = \frac{|x|}{R}$. ∎

[**Rk** Similar story if we replace $x \in \mathbb{R}$ by $z \in \mathbb{C}$: get convergence on a <u>disc</u>.
This "explains" why $\sum_{n=0}^{\infty} (-x^2)^n$ has $R=1$: $\frac{1}{1+z^2}$ has singularity at $z = i$]



Thm (Ratio Test)

1) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ converges.

2) If $\exists N$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ $\forall n \geq N$, then $\sum a_n$ diverges.

Pf See Rudin.

Thm If $\sum a_n = a$ and $\sum b_n = b$, then if $c_n = a_n + b_n$, $\sum c_n = a + b$.

Pf Exercise.

Def $\sum a_n$ <u>converges absolutely</u> if $\sum |a_n|$ converges.

Thm If $\sum a_n$ conv. abs. then $\sum a_n$ converges.

Pf Use $\left|\sum_{n=p}^{q} a_n\right| \leq \sum_{n=p}^{q} |a_n|$ and our criterion for series convergence above.

Abs. conv. series are the ones which may be treated liberally.

e.g.

Thm Suppose $\sum a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
Then $\sum_{n=0}^{\infty} c_n = AB$.

Pf Set $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$. $\beta_n = B_n - B$

Then
$$C_n = a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + \cdots + a_nb_0)$$
$$= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0$$
$$= A_nB + \underbrace{a_0\beta_n + \cdots + a_n\beta_0}_{\gamma_n}$$

Now we want to show $\gamma_n \to 0$.

Thus let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Since $\beta_n \to 0$, $\exists N$ s.t. $|\beta_n| \le \varepsilon$ for $n \ge N$.

Thus for $n \ge N$,
$$|\gamma_n| \le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0|$$
$$\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \alpha \cdot \varepsilon$$

Thus $\limsup_{n\to\infty} |\gamma_n| < \alpha \cdot \varepsilon$. But ε arbitrary. Thus indeed $|\gamma_n| \to 0$.

Def A <u>rearrangement</u> of a series $\sum a_n$
is a series $\sum a_{k_n}$
where $\{k_n\}$ is a sequence containing each $n \in \mathbb{N}$ once.

Rk Partial sums of the rearranged series may have little to do with the original one.
Also the limit may be different!

Ex
$$\underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{5/6} \underbrace{- \frac{1}{4} + \frac{1}{5}}_{<0} \underbrace{- \frac{1}{6} + \frac{1}{7}}_{<0} - \cdots \quad \text{converges to } s < \tfrac{5}{6}$$
$$\underbrace{\left(1 + \frac{1}{3}\right) - \frac{1}{2}}_{5/6} + \underbrace{\left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4}}_{>0} + \cdots \quad \text{" " } s > \tfrac{5}{6}$$

Thm If $\sum a_n$ is convergent but not absolutely convergent, and $-\infty \le \alpha \le \beta \le \infty$,
then $\exists$ a rearrangement $\sum a_{k_n}$ with partial sums s_n, such that $\limsup_{n\to\infty} s_n = \beta$, $\liminf_{n\to\infty} s_n = \alpha$.

Pf Rudin.

Thm If $\sum a_n$ is absolutely convergent, $\sum a_n = \alpha$, then all rearrangements of $\sum a_n$ also converge to α.

Pf Let $\sum a_{k_n}$ be a rearrangement of $\sum a_n$, with partial sums s_n'. Fix $\varepsilon > 0$.

$\exists N$ s.t. $\forall m,n \geq N$, $\sum_{k=m}^{n} |a_k| < \varepsilon/2$.

Pick p s.t. $\{1,2,\dots,N\} \subset \{n_1,\dots,n_p\}$. Then for $n>p$, $|s_n' - s_n| < \varepsilon/2$.

But $s_n \to \alpha$, ie $\forall n > N$, $|s_n - \alpha| < \varepsilon/2$.

Thus $\forall n > \max(N,p)$, $|s_n' - \alpha| < |s_n - s_n'| + |s_n - \alpha| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. ∎

Thm ("summation by parts") Let $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, $A_{-1} = 0$.

Then for $0 \leq p \leq q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Pf $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$ ∎

then break off first and last terms separately

Thm Suppose

1) Partial sums A_n of $\sum a_n$ form bounded sequence,
2) $b_0 \geq b_1 \geq \cdots$,
3) $b_n \to 0$.

Then $\sum a_n b_n$ converges.

Pf Pick M s.t. $|A_n| \leq M$ $\forall n$. Fix $\varepsilon > 0$.

$\exists N$ s.t. $b_N \leq \varepsilon/2M$.

For $N \leq p \leq q$,
$$\begin{aligned} \left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \qquad (\text{using } b_n - b_{n+1} \geq 0) \\ &= 2M b_p \\ &\leq 2M b_N \\ &\leq \varepsilon. \end{aligned}$$
∎

Cor Suppose $|c_1| \geq |c_2| \geq \cdots$, $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $\forall m$, and $c_n \to 0$.
Then $\sum_1 c_n$ converges.

Ex $\sum_1 \frac{(-1)^n}{n}$ converges (but not absolutely!)