M 365C
Fall 2013, Section 57465
Midterm 2

Each problem in **Part A** is worth **10 points**; **Part B** is worth **25 points**.

Part A. True or False. If true, sketch a proof in a few lines. If false, state a counterexample (in this case you do not have to prove that it is a counterexample.) You may use without proof anything that we proved in class or anything that is proved in Rudin chapters 1-6.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x \to 0} f(x) = 50$. Then there exists some neighborhood $N \subset \mathbb{R}$ of 0 such that for all $x \in N$, if $x \neq 0$, then $f(x) > 49$.

   True. Indeed, the definition of limit says that there exists some ϵ such that $0 < |x| < \epsilon \implies |f(x) - 50| < 1$; and if $|f(x) - 50| < 1$ them $f(x) > 49$.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, the set $E = \{x \in \mathbb{R} \,|\, f(x^2) \leq 7\}$ is closed in $\mathbb{R}$.

   True. Defining $g(x) = f(x^2)$, g is the composition of continuous functions and hence continuous; and defining $V = (-\infty, 7]$, V is a closed subset of $\mathbb{R}$. Then $E = g^{-1}(V)$ is also closed.

3. Suppose X and Y are metric spaces, and $\{a_n\}$ is a sequence in X. If $f : X \to Y$ is continuous, and the sequence $\{f(a_n)\}$ in Y is convergent, then $\{a_n\}$ is convergent.

   False. For example, take $\{a_n\}$ to be *any* diergent sequence in $\mathbb{R}$, e.g. $a_n = n$, and take f to be the constant function $f(x) = 0$. Then $\{f(a_n)\}$ is just the zero sequence, so it is convergent.

4. Suppose X and Y are metric spaces. If $f : X \to Y$ is continuous, and $E \subset X$, then $f(\bar{E}) = \overline{f(E)}$.

   False. For example, we could take $X = \mathbb{R}_+$ and $Y = \mathbb{R}$, $E = \mathbb{N}$, and $f(x) = 1/x$. Then $f(\bar{E})$ does not contain 0, but $\overline{f(E)}$ does. (We could also take $E = \mathbb{R}_+$ in fact.) Another example: take $X = E = \mathbb{R}$ and $Y = \mathbb{R}$, and $f(x) = e^x$. Then $f(E) = (0, \infty)$ and $\bar{E} = E$. Thus $f(\bar{E}) = (0, \infty)$, but $\overline{f(E)} = [0, \infty)$. Yet another example (in some sense the most fundamental): take $X = E = (0, 1)$ and $Y = [0, 1]$, and let $f(x) = x$. Then $\bar{E} = (0, 1)$, so $f(\bar{E}) = (0, 1)$, but $\overline{f(E)} = [0, 1]$.

Part B. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Define a new function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = f(x) + g(x)$. Prove carefully that h is continuous. Use only the definition of continuity.

Take any $p \in \mathbb{R}$ and suppose given some $\epsilon > 0$. Then, since f is continuous, there exists some δ' such that

$$|x - p| < \delta' \implies |f(x) - f(p)| < \epsilon/2$$

Similarly there exists some δ'' such that

$$|x - p| < \delta'' \implies |g(x) - g(p)| < \epsilon/2$$

Now take

$$\delta = \min(\delta', \delta'')$$

We then have by the triangle inequality

$$|x - p| < \delta \implies |h(x) - h(p)| < |f(x) - f(p)| + |g(x) - g(p)| < \epsilon$$

Thus h is continuous at p.