Applications of QFT to Geometry: Exercise Set 1

Exercise 1

Consider the zero-dimensional QFT with field space $\mathcal{C} = \mathbb{R}$, action $S : \mathcal{C} \to \mathbb{R}$ given by

$$S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4 \tag{0.1}$$

with $\mathrm{Re}(\lambda) \geq 0$, $m > 0$.

1. Compute the perturbative expansion of Z up to order λ^3, using Feynman diagrams. You should find $Z = \frac{\sqrt{2\pi}}{m}(1 - \frac{1}{8}\tilde{\lambda} + \frac{35}{384}\tilde{\lambda}^2 + \cdots)$ where $\tilde{\lambda} = \frac{\lambda}{m^2}$. (I drew the relevant diagrams in lecture, so this exercise just amounts to being sure you understand what I did.)
2. Compute the perturbative expansion of $\langle x^2 \rangle / Z$ up to order λ^3, using Feynman diagrams. You should find $\langle x^2 \rangle / Z = \frac{1}{m}(1 - \frac{1}{2}\tilde{\lambda} + \frac{2}{3}\tilde{\lambda}^2 - \frac{11}{8}\tilde{\lambda}^3 + \cdots)$. (I drew the diagrams up to order $\tilde{\lambda}^2$ in lecture.)
3. Compute the perturbative expansion of $\langle x^4 \rangle / Z$ up to order λ^2, using Feynman diagrams. You should find $\langle x^4 \rangle / Z = \frac{1}{m^2}(3 - 4\tilde{\lambda} + \frac{33}{4}\tilde{\lambda}^2 + \cdots)$.

Exercise 2

In the same zero-dimensional QFT considered in Exercise 1, consider holding m fixed and studying Z as a function of λ.

1. Show that $Z(\lambda)$ admits analytic continuation to a holomorphic function defined for $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$. (Hint: consider the change of variables $x \to x/\lambda^{1/4}$).
2. Show that $Z(\lambda)$ does *not* admit analytic continuation to a meromorphic function on the plane, or even on any neighborhood of $\lambda = 0$. (Hint: consider what happens when λ crosses $\mathbb{R}_-$.)
3. Show that $Z(\lambda)$ does not admit any convergent power series expansion around $\lambda = 0$.
4. Show that nevertheless the perturbative expansion of $Z(\lambda)$ is an *asymptotic* expansion for $Z(\lambda)$.

Exercise 3

Consider the supersymmetric zero-dimensional QFT with field space $\mathcal{C} = \mathbb{R}^{1|2}$ coordinatized by (x, ψ^1, ψ^2) and action $S : \mathcal{C} \to \mathbb{R}$ given by

$$S = \frac{1}{2}h'(x)^2 + h''(x)\psi^1\psi^2 \tag{0.2}$$

for some function $h : \mathbb{R} \to \mathbb{R}$.

1. In class I claimed that if $g = \rho''(x)\psi^2\psi^1 + h'(x)\rho'(x)$ for some function $\rho : \mathbb{R} \to \mathbb{R}$ then $\langle g \rangle = 0$. Verify this "directly," by first integrating over the fermions to reduce to an integral over x, and then using integration by parts.

2. I also claimed that $\langle g \rangle = -\frac{\partial}{\partial \lambda}|_{\lambda=0} \int_{\mathcal{C}} \mathrm{d}x \; \mathrm{d}\psi^1 \mathrm{d}\psi^2 \; e^{-(S+\lambda g)}$ as long as g/S is bounded, e.g. if g and S are polynomials with $\deg g \leq \deg S$. Prove this. (Hints to follow.)

Exercise 4

Fix a Riemannian manifold Y. Consider the one-dimensional QFT with field space $\mathcal{C} = \{\varphi : X \to Y\}$ and action $S = \int \frac{1}{2}\|\dot{\varphi}\|^2$, i.e. Lagrangian $L = \frac{1}{2}\|\dot{\varphi}\|^2 = \frac{1}{2} g_{ij}(\varphi(t))\dot{\varphi}^i(t)\dot{\varphi}^j(t)$.

1. Verify that the space $\mathcal{S}$ of classical solutions (critical locus of S) is the space of parameterized geodesics in Y. (I did this in class.) By considering "initial data" for the geodesics, identify $\mathcal{S}$ with TY or equivalently with T^*Y.
2. In class I wrote a general formula for the symplectic structure on $\mathcal{S}$ in cases where L depends only on the first derivatives of the fields:

$$\omega = \mathrm{d}\left[\frac{\partial L}{\partial \dot{\varphi}^i(t)}\right] \wedge \mathrm{d}\varphi^i(t).$$

   (Here we regard $\varphi^i(t)$ and $\frac{\partial L}{\partial \dot{\varphi}(t)}$ as functions on $\mathcal{S}$ just by restriction from $\mathcal{C}$.) Use this formula to show that $\mathcal{S}$ is symplectomorphic to T^*Y. (I stated this in class but didn't prove it.)