Overview

Suppose we are interested in studying <u>smooth manifolds</u> X.

One simple and powerful tool for this purpose: Fix Riem. metric g on X.

Let $\mathcal{H}_k = \{\omega \in \Omega^k(X) \mid \Delta\omega = 0\}$.

This is a <u>linear</u> equation over $\mathbb{R} \Rightarrow \mathcal{H}_k$ is a <u>vector space</u>.

If X is compact, $\mathcal{H}_k$ is finite-dimensional (b/c Δ is elliptic).

So, we get numbers $b_k(X) = \dim_{\mathbb{R}} \mathcal{H}_k$.

These turn out to depend only on <u>topology</u> of X, not the <u>metric</u>.

Now say X is a <u>4-manifold</u>.

A slight refinement of the above:

Fix an <u>orientation</u> on X. So, we have $\star: \Omega^2(X) \to \Omega^2(X)$. $\star^2 = 1$.

Thus $\mathcal{H}_2 = \underset{\substack{\uparrow \\ \star=1 \\ \text{"self-dual"}}}{\mathcal{H}_2^+} \oplus \underset{\substack{\uparrow \\ \star=-1 \\ \text{"anti-self-dual"}}}{\mathcal{H}_2^-}$ and we define $b_2^\pm(X) = \dim_{\mathbb{R}} \mathcal{H}_2^\pm$

$\left(\text{so } b_2(X) = b_2^+(X) + b_2^-(X)\right)$

Interesting and useful, but we want more.

Donaldson's idea: study solutions of <u>nonlinear</u> equations on M. (1983–1990 or so)

Fix a principal $SU(2)$ bundle E over M, and let D be a connection in E, $D \in \mathrm{Conn}(E)$. Curvature $F \in \Omega^2(X; \mathrm{ad}\, P)$.

$$\left[\begin{array}{lll} \text{Locally:} & D = d + A & A \in \Omega^1(X; su(2)) \\ & = d + A_i T_i & A_i \in \Omega^1(X) \\ & F = dA + A \wedge A & F \in \Omega^2(X; su(2)) \\ & = \underbrace{\left(dA_i + \varepsilon_{ijk} A_j A_k\right)}_{F_i} T_i & F_i \in \Omega^2(X) \end{array}\right]$$

Natural action of "gauge group" $\mathcal{G} = \{\text{sections of } \mathrm{Aut}(E)\}$ on $\mathrm{Conn}(E)$ preserving F.

Now, let $\mathcal{M} = \{ D \in \mathrm{Conn}(E) \mid F = -\star F \} / \mathcal{G}$

It's not a vector space, but does have manifold structure (at least away from "reducible connections"). The linearization is elliptic, so if X is compact, $\mathcal{M}(E)$ is <u>finite-dimensional</u>.

$$\left[\dim_{\mathbb{R}} \mathcal{M} = 8c_2(E) - 3(1 - b_1(X) + b_2^+(X)) \right]$$

And, one can construct some natural top-degree <u>forms</u> ω_α on $\mathcal{M}$, labeled by "observables" α. (related to homology of X)

Then, define <u>Donaldson invariants</u> $\langle \alpha \rangle = \int_{\mathcal{M}} \omega_\alpha$

(Technically difficult — especially b/c we need to <u>compactify</u> $\mathcal{M}$)

These invariants depend only on the <u>smooth structure</u> of X, not on the metric (as long as $b_2^+(X) > 1$).

They turned out to be powerful tools!

But not easy to compute, define, use...

In 1988 Witten made a remarkable discovery (following impt work of Floer): the Donaldson invariants have a natural interpretation as a part of

$\mathcal{N}=2$ supersymmetric $SU(2)$ gauge theory in 4 dimensions

(A close cousin of the kind of theory that describes our actual Universe!)

Very roughly, we imagine taking X to be our "spacetime" and performing some "experimental measurements." This (as we'll see) amounts to studying integrals over <u>infinite-dimensional</u> space $\mathcal{C}$. But in this case, they actually <u>reduce</u>: $\int_{\mathcal{C}} = \int_{\mathcal{M}}$.

Localization.

Exciting, but doesn't lead to any spectacular progress in <u>computing</u> the invariants.

The real progress came in 1995 when Seiberg and Witten made a fundamental discovery about the physics of $\mathcal{N}=2$ supersymmetric gauge theory in 4 dimensions.

An analogy: Suppose you want to study water waves.

One approach: water $\supset$ molecules $\supset$ atoms $\supset$ electrons, quarks, ...
So, use equations of QCD (and a big computer).

Much better approach: use the equations of hydrodynamics.
All complicated details of short distance physics are replaced by finite # of "effective" parameters: density, viscosity. They could in principle be computed, but this would be very hard.

This "effective theory" is far simpler, though not as powerful as the original one — it can't tell you e.g. what happens if you shoot a high-power laser at the water. Roughly it knows the answer to all "low energy questions."

Seiberg and Witten solved this pb completely for $\mathcal{N}=2$ SUSY g.t. in 4 dim:
They showed that the physics a low energy observer sees is governed by abelian gauge theory (coupled to matter). A very strong statement!

So, we can try to compute the results of experiments using this "effective" abelian theory.
(= Donaldson invariants)

As before, the answer turns out to localize on some simple equations:

$$\tilde{\mathcal{M}} = \left\{ (D,\psi) \;\middle|\; \begin{matrix} F + \star F = q(\psi,\bar\psi) \\ \not{D}\psi = 0 \end{matrix} \right\} \Big/ \mathcal{G}$$

Here F is the curvature of a connection in a $U(1)$ bundle E
ψ is a section of a spinor bundle $S \simeq \bar{S}$
q is the quadratic map $S \otimes S \longrightarrow \Lambda^2_+(T^*X)$

So, find that Donaldson inv'ts can be obtained by integrals over $\widetilde{\mathcal{M}}$ instead of $\mathcal{M}$!

Moreover, $\widetilde{\mathcal{M}}$ is compact! Much easier. (Like hydrodynamics.)

This led to a revolution in 4-mfd topology...

The plan of the course is to fill in as many details of the above story as we can. We'll emphasize the physical reason why Donaldson and Seiberg-Witten are related, not so much the application of either theory to 4-mfd topology (but might get "pinch hitters" to discuss the latter).

Begin in 0 dim, then 1, then try to jump to 4...

References: posted on webpage. Short term, the most relevant one should be the Clay Math "Mirror Symmetry" book.

Assignments: do 1 exercise per sheet, or, write a short paper.
All due May 9.