Last time:

Donaldson	Seiberg-Witten
SDYM equations	Seiberg-Witten equations
high energy	low energy
short distance	long distance
hard	easy

Now: let's try to understand the relation between the two...

First, via a zero-dimensional model.

QFT in zero dimensions

In general, QFT on X involves $\int$ over some function spaces $\mathcal{C}$ on X.
Infinite-dimensional, with one exception: $X = \text{point}$.
So, let's take $\mathcal{C} = \mathbb{R}$.

Define $S : \mathcal{C} \to \mathbb{R}$ by $S(x) = \frac{m}{2}x^2 + \frac{\lambda}{4!}x^4$. Say $\mathrm{Re}(\lambda) \geq 0$, $m > 0$.

Define partition function:

$$Z = \int_{-\infty}^{\infty} dx\, e^{-S(x)}$$

and for any polynomial $f : \mathcal{C} \to \mathbb{R}$ (observable), define expectation value

$$\langle f \rangle = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-S(x)} \qquad (\text{So } Z = \langle 1 \rangle).$$

How do we compute Z? For $\lambda = 0$, it's easy: $Z_0 = Z(\lambda=0) = \sqrt{\frac{2\pi}{m}}$

So, try expanding around $\lambda = 0$ (perturbation expansion):

$$Z = \int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{x^{4n}}{n!} e^{-\frac{m}{2}x^2}$$

$$\text{“}=\text{”} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dx\, x^{4n} e^{-\frac{m}{2}x^2}$$

$$= \sqrt{\frac{2\pi}{m}} \left(1 - \frac{1}{8}\tilde{\lambda} + \frac{35}{384}\tilde{\lambda}^2 + \cdots + 1390.1\,\tilde{\lambda}^{10} + \cdots \right) \qquad \left[\tilde{\lambda} = \frac{\lambda}{m^2} \right]$$

Convergence looks doubtful.

Indeed, this series diverges for <u>all λ</u>, even though Z exists for $\mathrm{Re}(\lambda) \geq 0$.

We could not have hoped for more — $f(\lambda)$ has <u>analytic continuation</u> to $\lambda \in \mathbb{C} \setminus \{0\}$ but with <u>essential singularity</u> at $\lambda = 0$. Such a f^n can't have convergent series $\exp^n$ there.

The series still has a meaning tho: it is an <u>asymptotic series</u> for $Z(\lambda)$ as $\lambda \to 0^+$.

<u>Def</u> $\sum_{n=0}^{\infty} a_n \lambda^n$ is an <u>asymptotic series</u> for $f(\lambda)$ as $\lambda \to 0^+$ if $\forall N$,

$$\lim_{\lambda \to 0^+} \frac{\left| f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n \right|}{\lambda^N} = 0 .$$

This is useful information about f!

Similarly define perturbation expansions for $\langle f \rangle$.

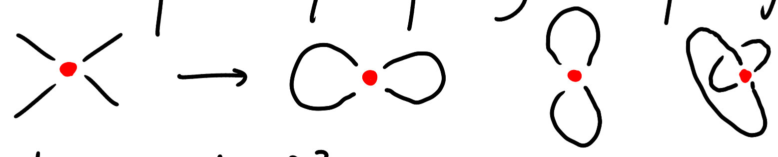
How to really compute the coefficients in $Z = \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!} \right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dx \; x^{4n} e^{-\frac{m}{2}x^2}$:

Use $\int dx \, x^{2k} e^{-\frac{m}{2}x^2} = \sqrt{\frac{2\pi}{m}} \times \frac{1}{m^k} \times \frac{(2k)!}{k!\,2^k}$ (pf: $\int$ by parts, induction)

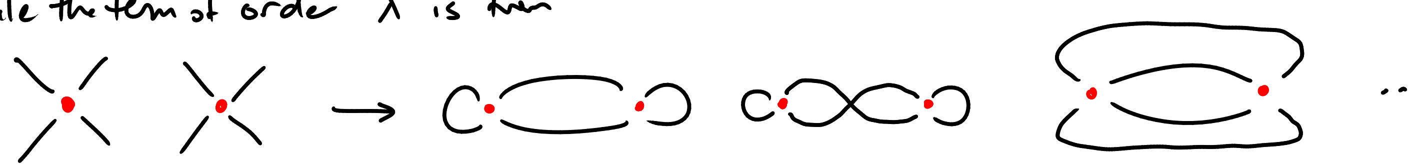
Now $\frac{(2k)!}{k!\,2^k}$ is the # of ways to pair up $2k$ objects. (pf: induction)

So, a combinatorial interpretation:

each $-\lambda$ in the expansion is coming with 4 "objects" (half-edges), we consider all possible ways of pairing them up. e.g. the term of order λ comes from



while the term of order λ^2 is from



Let $D_n = \{\text{diagrams with } n \text{ vertices}\}$

then D_n is naturally acted on by $G_n = (S_4)^n \rtimes S_n$. $|G_n| = n!\,(4!)^n$.

So, we can rewrite our expansion as

$$Z = \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \frac{1}{m^{2n}} (-\lambda)^n \frac{|D_n|}{|G_n|}$$

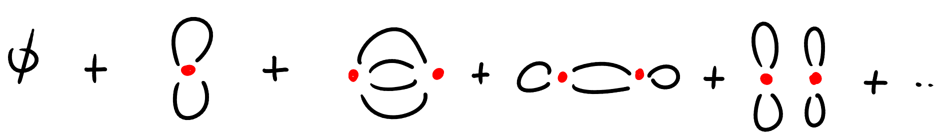
Let O_n be the set of orbits of G_n action on D_n.

Then (orbit-stabilizer thm) $\dfrac{|D_n|}{|G_n|} = \displaystyle\sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}$ $\quad$ ($\mathrm{Aut}\,\Gamma$ = the stabilizer in D_n of any element in Γ)

So finally, $Z = \sqrt{\dfrac{2\pi}{m}} \displaystyle\sum_{n=0}^{\infty} \frac{1}{m^{2n}} (-\lambda)^n \sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}$

$$= \sqrt{\frac{2\pi}{m}} \sum_{\Gamma} \frac{(-\lambda)^{|v(\Gamma)|}}{m^{|e(\Gamma)|}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|}$$

Concretely,

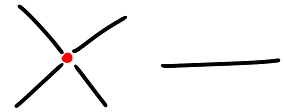


$$\frac{Z}{Z(\lambda=0)} = 1 + \frac{-\lambda}{8m^2} + \frac{\lambda^2}{48m^4} + \frac{\lambda^2}{16m^4} + \frac{\lambda^2}{128m^4} + O(\lambda^3)$$

Last time: "Feynman rules" for computing perturbative expansion of $Z=\int_{-\infty}^{\infty} dx\, e^{-S}$ where $S(x)=\frac{1}{2}mx^2+\frac{\lambda}{4!}x^4$.

$$\frac{Z}{Z_0}=\sum_{\Gamma} wt(\Gamma)$$

where each Γ is a graph w/ only 4-valent vertices, $wt(\Gamma)$ is built by multiplying ingredients



$-\lambda$ $\qquad \frac{1}{m}$

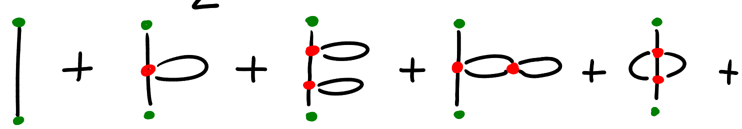
and dividing by $|Aut(\Gamma)|$.

Some variants:

① $$\log\left(\frac{Z}{Z_0}\right)=\sum_{\Gamma \text{ connected, nonempty}} wt(\Gamma)$$

② To compute $\langle x^n\rangle$ we use graphs w/ n 1-valent vertices. Automorphisms fix these vertices. For $\frac{\langle x^n\rangle}{Z}$ we take graphs s.t. no component has no 1-valent vertices.

E.g. $\frac{\langle x^2\rangle}{Z}=$



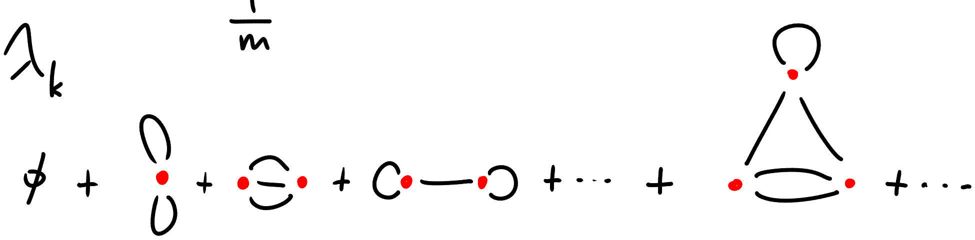
$$\frac{\langle x^2\rangle}{Z}=\frac{1}{m}-\frac{\lambda}{2m^3}+\frac{\lambda^2}{4m^5}+\frac{\lambda^2}{4m^5}+\frac{\lambda^2}{6m^5}+O(\lambda^3)$$

③ If more generally we took $S=\frac{m}{2}x^2+\sum_{k=3}^{\infty}\frac{\lambda_k}{k!}x^k$, then we allow vertices of all valences and Feynman rules:

valence k

$-\lambda_k$ $\qquad \frac{1}{m}$

E.g. $\frac{Z}{Z_0}=$



$$\frac{Z}{Z_0}=1-\frac{\lambda_4}{8m^2}+\frac{\lambda_3^2}{12m^3}+\frac{\lambda_3^2}{8m^3}+\cdots-\frac{\lambda_3^2\lambda_4}{8}+\cdots$$

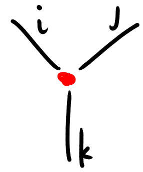
④ Now say $\mathcal{C} = \mathbb{R}^n$ instead of $\mathbb{R}$.

$$S = \frac{1}{2} X^i M_{ij} X^j + \frac{C_{ijk}}{6} X^i X^j X^k \qquad \text{(Einstein summation convention)}$$

then $$Z_0 = \int d\vec{X}\; e^{-\frac{1}{2} X^i M_{ij} X^j} = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}$$

Feynman rules:

$= (M^{-1})^{ij}$



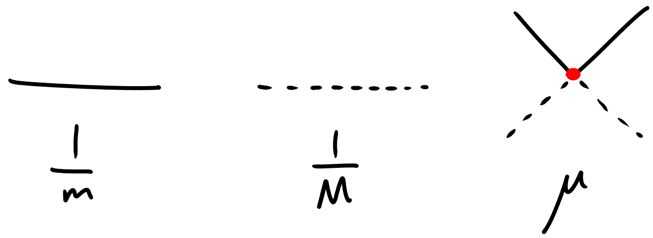
$= -C_{ijk}$

A coupled system

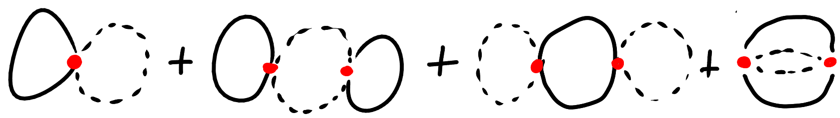
Now say $\mathcal{C} = \mathbb{R}^2$ and $S(x,y) = \frac{m}{2}x^2 + \frac{M}{2}y^2 + \frac{\mu}{4}x^2y^2$

For $f: \mathbb{R}^2 \to \mathbb{R}$ polynomial, $\langle f \rangle = \int_{\mathbb{R}^2} dx\,dy\; f(x,y)\, e^{-S(x,y)}$

Feynman rules:

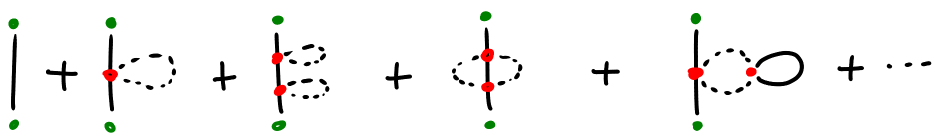


$$\text{e.g. } \log\left[\frac{Z}{Z_0}\right] =$$



$$= -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + \mathcal{O}(\mu^3)$$

$$\frac{\langle x^2 \rangle}{Z} =$$



$$= \frac{1}{m} - \frac{\mu}{2m^2M} + \frac{\mu^2}{4m^3M^2} + \frac{\mu^2}{2m^3M^2} + \frac{\mu^2}{4m^3M^2} + \mathcal{O}(\mu^3)$$

Integrating out

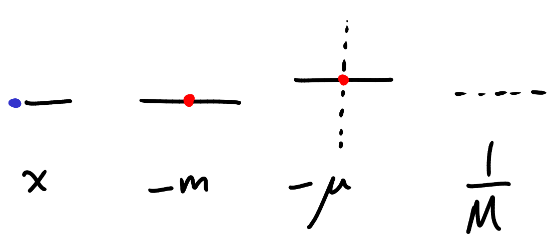
We can also think of computing these by Fubini's thm: integrate over y first.

Define $S_{\text{eff}}(x) = -\log\left[\int_{-\infty}^{\infty} dy\; e^{-S(x,y)}\right]$ ("effective action")

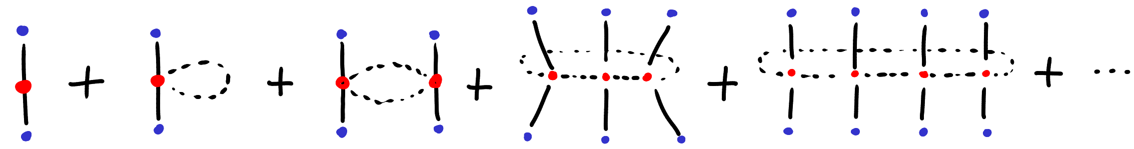
then $\langle f(x) \rangle = \int dx\; f(x)\; e^{-S_{\text{eff}}(x)}$

To compute the perturbative expansion of $S_{\text{eff}}(x)$: sum over diagrams where all x lines terminate on 1-valent vertices. (vertices not fixed by automorphisms)

Feynman rules:



$$-S_{\text{eff}}(x) =$$



$$= -\frac{m}{2}x^2 - \frac{\mu}{4M}x^2 + \frac{\mu^2}{16M^2}x^4 - \frac{\mu^3}{48M^3}x^6 + \frac{\mu^4}{128M^4}x^8 + \cdots$$

(general term: $\left(-\frac{\mu}{M}\right)^k \frac{1}{2^{k+1}k}\, x^{2k}$)

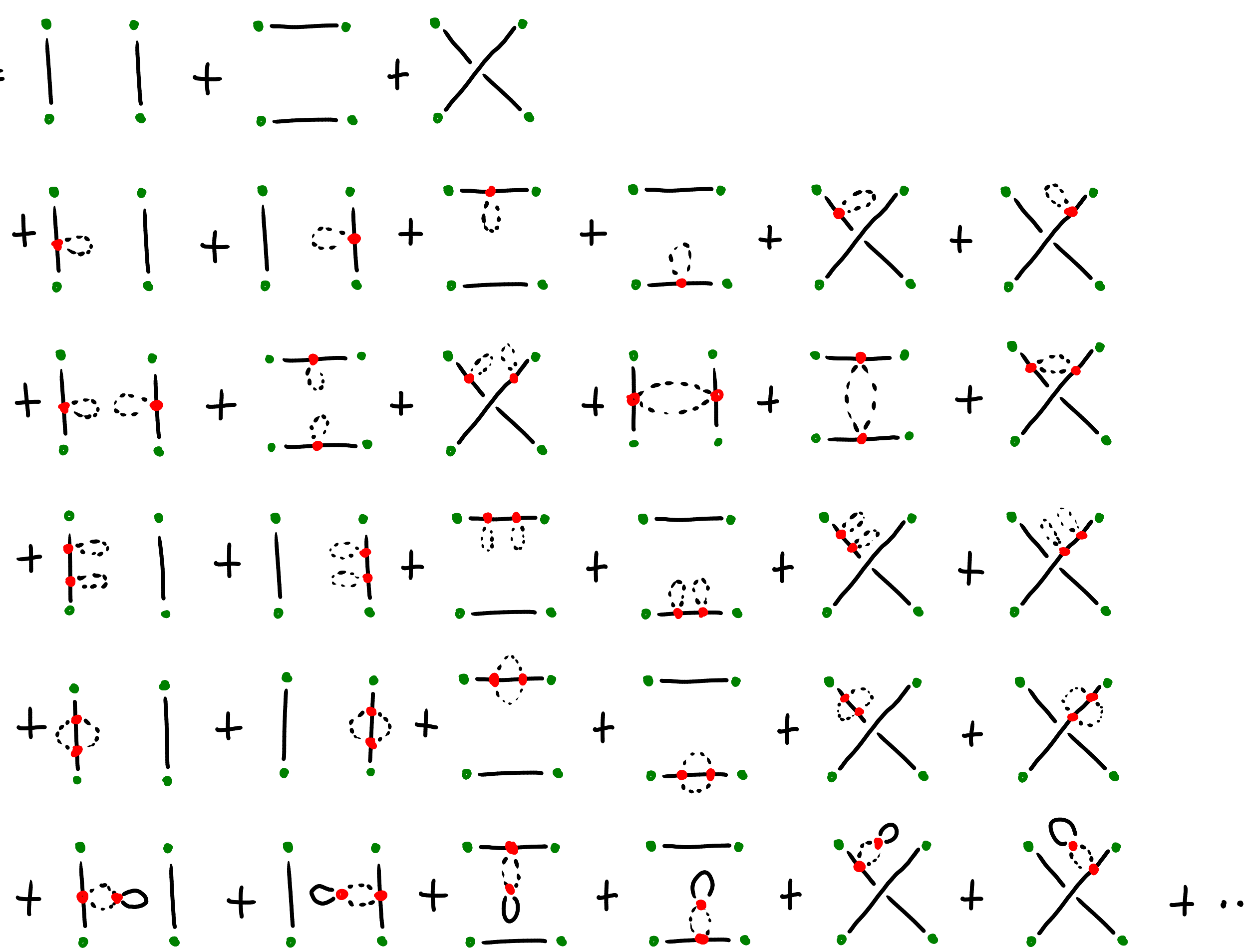
Interpretation: the effect of integrating out y is to <u>shift</u> the quadratic term in $S(x)$ $\left(m \to m + \frac{\mu}{2M}\right)$

and also <u>generate</u> higher-order interaction terms, which encapsulate the effect of all the diagrams involving y.

Working with $S_{\text{eff}}(x)$ is <u>easier</u> than working with $S(x,y)$ — as long as we only want to compute $\langle 1 \rangle$ or $\langle f(x) \rangle$, not $\langle f(x,y) \rangle$...

So e.g. using $S(x,y)$ we get

$$\frac{\langle x^4 \rangle}{Z} =$$

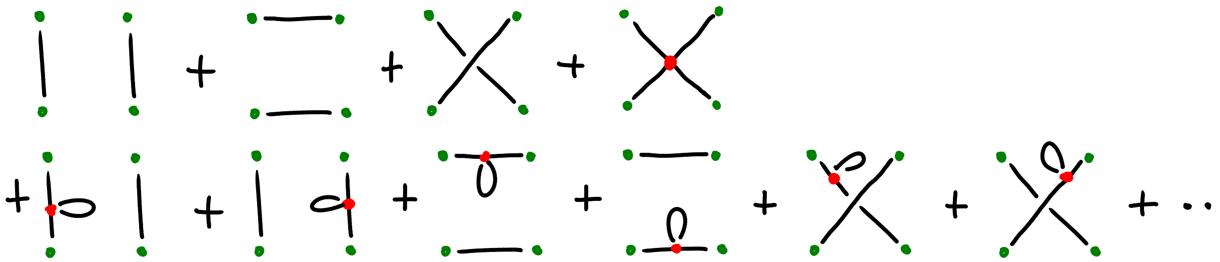


$$= \frac{3}{m^2} - \frac{3\mu}{m^3 M} + \frac{33\mu^2}{4m^4M^2} + \mathcal{O}(\mu^3)$$

while using $S_{eff}(x) = \frac{m_{eff}}{2}x^2 + \frac{\lambda_4}{4!}x^4 + \frac{\lambda_6}{6!}x^6 + \cdots$

where $m_{eff} = m + \frac{\mu}{2M}$, $\lambda_4 = -\frac{3}{2}\mu^2$, $\lambda_6 = 15\mu^3$, ...

$$\frac{\langle x^4\rangle}{2} =$$



$$= \frac{3}{m_{eff}^2} - \frac{4\lambda_4}{m_{eff}^4} + \mathcal{O}(\mu^3)$$

The two computations indeed agree, but using $S_{eff}(x)$ is a lot quicker...