Duistermaat-Heckman formula

Say M compact symplectic
with $U(1)$ action generated by Hamiltonian H, w/ vector field $Y = \omega^{-1}(dH)$

Then, consider a supermanifold $\mathcal{E} = \Pi TM$ obtained by making the fibers of TM odd. (Local coords: x^i even, ψ^i odd)
$\mathrm{Fun}(\mathcal{E}) = \Omega^*(M)$.

Let $S = H + \omega_{ij}\psi^i\psi^j$

invariant under $V = \psi^i \frac{\partial}{\partial x^i} + y^j \frac{\partial}{\partial \psi^j}$ $\qquad \{V, V\} = \psi^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial \psi^j} + y^i \frac{\partial}{\partial x^i}$

V acts by $d + \iota_Y$ on $\mathrm{Fun}(\mathcal{E})$ $\qquad \{V, V\}$ then acts as $\mathcal{L}_Y$ (Cartan's formula)

Now, $Z = \int_{\mathcal{E}} \prod dx^i d\psi^i \, e^{-S} = \frac{1}{n!}\int_M \omega^n e^{-H}$

Our heuristic says: should be able to evaluate this by localization to the fixed points of V and Gaussian approx. around each one.
This would lead to the result:

$$Z = \frac{(2\pi)^{n/2}}{n!}\sum_{x \text{ fixed}} e^{-H(x)} \frac{\mathrm{Pf}(\omega_{ij})}{\sqrt{\det \partial_i\partial_j H}} \quad \left[\sim \sum_{x \text{ fixed}} \frac{e^{-H(x)}}{\prod_i wt_i(x)} \text{ where } wt_i \text{ denote the weights of the } U(1) \text{ action on } (T_x M)_{\mathbb{C}}\right]$$

Justified in this case, by a procedure similar to what we did in previous case:
pick $U(1)$ inv. metric g on M and take $\sigma = g_{ij}\psi^i y^j = \psi^i Y_i$. Then $V(\sigma) = g(Y,Y) + \psi^i\psi^k \frac{\partial}{\partial x^k} Y_i$

So: $V(\sigma) = 0$ only at fixed pts, > 0 otherwise;
$V^2(\sigma) = 0$ (b/c σ is $U(1)$ invariant)

Replace $S \to S + \lambda V(\sigma)$, note Z is indep of λ (V divergence-free), then take $\lambda \to \infty$.

Evidently this should give localiz^n to fixed points.

Surprisingly, the extra contribs. to quadratic terms from $V(\sigma)$ <u>cancel</u> one another, so we still just recover the original Gaussian approx. determined by S.

This case is a bit simpler than the previous one, because we had $V^2(\sigma)=0$ so the actions S for all λ were inv^t under the <u>same</u> V. In this setting Schwarz-Zaboronsky actually proved a general localization thm.

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