Standard example: harmonic oscillator. Take $M = \mathbb{R}$, $V = \frac{1}{2}x^2\omega^2$.

Then $\mathcal{H} = L^2(\mathbb{R})$, $H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2\omega^2$ (densely defined)

Eigenfunctions:
$$\psi_0(x) = e^{-\omega x^2/2}$$
$$\psi_1(x) = x e^{-\omega x^2/2}$$
$$\psi_2(x) = \left(x^2 - \frac{1}{2\omega}\right) e^{-\omega x^2/2}$$
$$\vdots$$

$$\psi_n(x) = H_n(x)\, e^{-\omega x^2/2}$$

"Hermite polynomials"

[Can get them as $|\psi_n\rangle = (a^\dagger)^n |\psi_0\rangle$ where $a^\dagger = \frac{\partial}{\partial x} - \omega x$ ("creation operator") which obeys $[a^\dagger, H] = \omega a^\dagger$]

$$H\psi_n = \left(\tfrac{1}{2} + n\right)\omega\, \psi_n$$

So in particular,
$$\mathrm{Tr}\, e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\omega\beta(n+\frac{1}{2})} = \frac{1}{2\sinh(\omega\beta/2)}$$

Path integral version: $X = S^1_\beta$ of length β, $Y = \mathbb{R}$, $V(x) = \frac{1}{2}x^2 \Rightarrow$

So $\mathcal{C} = \mathrm{Map}(S^1, \mathbb{R})$ and $S: \mathcal{C} \to \mathbb{R}$ is

$$S[x(t)] = \int_{S^1_\beta} dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2\omega^2\right) = \frac{1}{2}\int_{S^1_\beta} dt\; x\left(-\frac{d^2}{dt^2} + \omega^2\right)x$$

$$Z = \int_{\mathcal{C}} \mathcal{D}x\, e^{-S[x]}$$

This is a Gaussian integral — but now over infinitely many variables!

Diagonalize the operator $-\frac{d^2}{dt^2} + \omega^2$:

$$x(t) = \frac{c}{\sqrt{\beta}} + \sum_{n=1}^{\infty} a_n\sqrt{\frac{2}{\beta}}\sin\left(\frac{2\pi n}{\beta}t\right) + b_n\sqrt{\frac{2}{\beta}}\cos\left(\frac{2\pi n}{\beta}t\right)$$

$$\mathcal{D}x = C\cdot\frac{dc}{\sqrt{2\pi}}\prod_{n=1}^{\infty}\frac{da_n}{\sqrt{2\pi}}\frac{db_n}{\sqrt{2\pi}} \qquad S[x] = \frac{1}{2}\left[\omega^2c^2 + \sum_{n=1}^{\infty}\left(\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2\right)\left(a_n^2 + b_n^2\right)\right]$$

$$\longrightarrow \int_{\mathcal{C}} \mathcal{D}x\, e^{-S[x]} = \frac{C}{\omega}\prod_{n=1}^{\infty}\left[\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2\right]^{-1}$$

$$\Rightarrow \quad Z = \frac{C}{\omega} \prod_{n=1}^{\infty} \left[\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2 \right]^{-1}$$

This is tricky to interpret: the product is divergent (to 0), and also, C may be infinite.

One approach to understanding it:

We could imagine regulating the theory (e.g. by discretization). This should amount to cutting off the product at some finite N. So, we may write

$$Z = \lim_{N\to\infty} Z_N, \quad Z_N = \frac{C_N}{\omega} \prod_{n=1}^{N} \left[\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2 \right]^{-1}$$

In particular, this means

$$\frac{Z(\omega)}{Z(\omega')} = \lim_{N\to\infty} \frac{\omega'}{\omega} \prod_{n=1}^{\infty} \frac{\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2}{(\omega')^2 + \left(\frac{2\pi n}{\beta}\right)^2} \overset{\text{use product formula for sinh}}{=} \frac{\sinh\left(\frac{\omega\beta}{2}\right)}{\sinh\left(\frac{\omega'\beta}{2}\right)}$$

which is enough to tell us

$$Z = \frac{\alpha}{\sinh\left(\frac{\beta\omega}{2}\right)} \qquad \text{for some } \alpha.$$

In principle α could depend on β. But by change of variables $t = \beta t'$, $x = \frac{1}{\sqrt{\beta}} x'$ formally see Z depends only on $\beta\omega$, not on the two separately. So α should be a single "universal" constant.

By comparing with the operator formalism we see that we should put $\alpha = \frac{1}{2}$.

So even without being very precise about the details of the regulator here, we got the right answer up to one undetermined constant. Easier to determine that constant some other way. (NB, we could have gotten it by studying the limit $\beta \to \infty$ where $\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}$ is dominated by the ground state; don't need to know the whole spectrum of $\mathcal{H}$.)

There is a "magic trick" that even gets the constant right: zeta regularization