EFT in quantum mechanics:

Consider a system of 2 particles interacting:

QM with $Y = \mathbb{R}^2$, $V = \frac{1}{2}x^2 + \frac{1}{2}\omega^2 y^2 + \frac{\mu}{4}x^2y^2$ $\quad S[x,y] = \int dt\ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V(x,y)$

Classical equations: $\ddot{x} = x + \frac{1}{2}\mu x y^2$
$\ddot{y} = \omega y + \frac{1}{2}\mu x^2 y$

Remember, we're in Euclidean signature! To think more physically about this, continue $t \to it$.
Then, if $\omega \gg 1$, y is "oscillating faster" than x $\Rightarrow$ details of y shouldn't matter for dynamics of x, at least if $\dot{x}, \ddot{x}, \dots$ small enough (say ~ 1, which is their characteristic value...)

Suppose we are interested in expectation values involving only x.

e.g. in $$\underbrace{\langle x(0)\,x(t)\rangle}_{\frac{\mathrm{Tr}_{\mathcal{H}}\, e^{-Ht} x\, e^{-H(\beta-t)} x}{\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}}} = \frac{1}{Z}\int_{\mathcal{E}_{S^1_\beta}} \mathcal{D}x\,\mathcal{D}y\ \ x(0)x(t)\ e^{-S[x,y]}$$

An asymptotic series for this in μ can be evaluated by Feynman diagrams:
an extension of what we did in 0 dimensions.

Basic ingredient: Green's functions D_x, D_y on the circle

$$[\partial_t^2 - 1]\, D_x(t) = \delta(t) \qquad D_x(t) = \frac{1}{2}\sum_{n\in\mathbb{Z}}{}' e^{-|t+n\beta|}$$

$$[\partial_t^2 - \omega^2]\, D_y(t) = \delta(t) \qquad D_y(t) = \frac{1}{2\omega}\sum_{n\in\mathbb{Z}}{}' e^{-\omega|t+n\beta|}$$

$\langle x(0)x(t)\rangle =$ 

$$= D_x(t) + \frac{\mu}{2}\int_{S^1_\beta} dt'\ D_x(t')D_x(t-t')D_y(0) + \frac{\mu^2}{2}\iint dt\,dt'\ D_y(t'-t'')^2 \times D_x(t-t')D_x(t'-t'')D_x(t'')$$
$$+ \frac{\mu^2}{4}\iint dt'dt''\ D_y(0)^2 D_x(t')D_x(t''-t')D_x(t-t'') + O(\mu^3)$$

Lots of diagrams — like we saw in 0-dimensional QFT. Each diagram gives *finite* result.
These kinds of computation are $\simeq$ to what is usually called "perturbation theory" in QM.

Now, how about the "effective field theory" point of view?

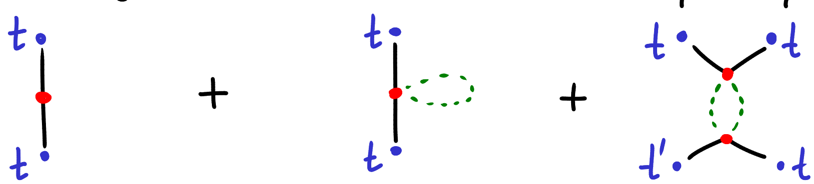
Formally speaking we could compute such expectation values by *first* "integrating over $y(t)$":

$$S_{\text{eff}}(x) = -\log \int \mathcal{D}y\; e^{-S[x,y]}$$

and then using the action $S_{\text{eff}}(x)$ for all further computations.

But, there is no reason why $S_{\text{eff}}(x)$ should be of the form $S_{\text{eff}}(x) = \int dt\; L_{\text{eff}}[x(t), \dot{x}(t)]$!

Indeed, by the same diagram rules we used in 0 dim (or by directly *expanding* in powers of x),

$S_{\text{eff}}(x) =$ 

$+ \cdots$

$$\int dt\; \tfrac{1}{2}\dot{x}(t)^2 + \tfrac{1}{2}x(t)^2 + \frac{\mu}{2}\int dt\; x(t)^2 D_y(0) + \underbrace{\frac{\mu^2}{2}\int dt\,dt'\; x(t)^2 x(t')^2 D_y(t-t')^2}_{\text{non-local interaction!}} + \cdots$$

This non-locality looks unpleasant. But all hope is not lost.

$D_y(t-t')$ decays *exponentially* away from the diagonal. *Morally*, then, the non-locality is "exponentially small."

This suggests *expanding* the non-local interaction:

$$\int dt\,dt'\; x(t)^2 x(t')^2 D_y(t-t')^2$$

$$= \int dt\,dt'\; \left[x(t)^4 + 2x(t)^3\dot{x}(t)\,(t'-t) + \left[x(t)^2\dot{x}(t)^2 + \tfrac{1}{2}x(t)^3\ddot{x}(t)\right](t'-t)^2 + \cdots\right] D_y(t-t')^2$$

and perform the integrals over t' to get

$$= \int dt\;\; c_1 x^4 + \frac{c_2}{\omega^2}\left(x^2\dot{x}^2 + \tfrac{1}{2}x^3\ddot{x}\right) + \cdots$$

An infinite series of higher-derivative interactions, suppressed by powers of ω.
So, in the limit of small $\dot{x}, \ddot{x}, \ldots$ and large ω, this gives a systematic expansion.
If we keep only finitely many terms, the theory looks <u>local</u> after all —
matching our original intuition.

If we want to compute e.g. $\langle x(0)\, x(t) \rangle$ for $\omega t \gg 1$, this approximation works well.

It <u>breaks down</u> if we try to study things involving $\dot{x} \sim \omega$.
(In particular, if $\omega \to 0$.)

This is a <u>model</u> for something that happens in Donaldson theory...