Clifford algebras

Say V real vector space w/ + def bilinear form.

Tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ w/ obvious product

Clifford algebra $C\ell(V) = T(V) \Big/ \langle v \otimes v - \frac{1}{2}\|v\|^2 \cdot 1 \rangle$ over $\mathbb{R}$

Facts about $C\ell(V)$: [see e.g. Morgan book]

- Has a $\mathbb{Z}_2$-grading $C\ell(V) = C\ell^0(V) \oplus C\ell^1(V)$ inherited from $\mathbb{Z}$-grading of $T(V)$

- Taking an orthonormal basis $\{e_i\}$ for V, $C\ell(V)$ is generated by the e_i with the relation $e_i e_j + e_j e_i = \delta_{ij}$

- Basis of unordered monomials in the e_i, using each e_i at most once $\Longrightarrow \dim_{\mathbb{C}} C\ell(V) = 2^{\dim V}$

- If $\dim V$ even, $C\ell(V)$ has a unique f-d complex irrep $S(V)$.
  $\dim S(V) = 2^{\frac{1}{2}\dim V}$. Under $C\ell^0(V)$, have decomp. $S(V) = S^0(V) \oplus S^1(V)$

[e.g. if $\dim V = 2$, $C\ell(V) = \{1, e_1, e_2, e_1 e_2\}$, $\dim S(V) = 2$;

e_1 acts by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ e_2 acts by $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$]

- Define $Pin(V)$ = subgp of $C\ell(V)$ gen. by $v \in V$ with $\|v\|^2 = 1$
  $Spin(V) = Pin(V) \cap C\ell^0(V)$

  $Spin(V)$ acts on $V \subset C\ell(V)$ by conjugation, this gives $\underbrace{Spin(V) \to SO(V)}_{\text{double cover}}$

- $S^0(V)$, $S^1(V)$ are <u>irreps</u> of $Spin(V)$ [but <u>don't</u> descend to $SO(V)$]