Supersymmetric quantum mechanics

Refs: Witten "Constraints on Supersymmetry Breaking"
Alvarez-Gaumé "Supersymmetry and the index theorem"

We've seen that given a Riem. mfd Y, there is a corresponding 1-d QFT that computes the spectrum of Δ on Y $\quad \left(Z_{S^1_\beta} = \mathrm{Tr}\, e^{-\beta\Delta}\right)$

This depends on "every little detail" of Y, morally.
It's very far from being topological.

Fortunately, there is something else we can do: add fermions!

A new QFT: $\mathcal{E} = \left\{ (\varphi,\psi) : \begin{array}{ll} \varphi : X \to Y & \text{even} \\ \psi \in \Pi[\varphi^*(TY)] & \text{odd} \end{array} \right\}$ $\left[= \mathrm{Map}(X \times \mathbb{R}^{0|1}, Y) \right]$

$$S = \int dt\, \tfrac{1}{2}\left(\|\dot\varphi\|^2 + \langle \psi, \nabla_t \psi \rangle \right) \qquad [\nabla_t = \text{pulled-back Levi-Civita conn}]$$

As before, this is invariant under a vector field W representing time translations: $\begin{bmatrix} \delta\varphi = \varepsilon\dot\varphi \\ \delta\psi = \varepsilon\dot\psi \end{bmatrix}$

More interesting, it is also inv't under an odd vector field V: $\begin{bmatrix} \delta\varphi = \varepsilon\psi \\ \delta\psi = -\varepsilon\dot\varphi \end{bmatrix}$

$\{V,V\} = 2W$ so these two v.f. together give an action of $\mathbb{R}^{1|1}$

[Checking $VS = 0$:

$$S = \int \tfrac{1}{2} g_{IJ}(\varphi)\left(\dot\varphi^I \dot\varphi^J + \psi^I \dot\psi^J + \psi^I \dot\varphi^K \Gamma^J_{KL}(\varphi)\psi^L \right)$$

$$\delta S = \tfrac{1}{2}\int g_{IJ,M}(\varphi)\left(\varepsilon\psi^M \dot\varphi^I \dot\varphi^J + \varepsilon\psi^M \psi^I \dot\psi^J + \varepsilon\psi^M \psi^I \dot\varphi^K \Gamma^J_{KL}(\varphi)\psi^L \right)$$
$$+ g_{IJ}(\varphi)\Big(2\varepsilon\dot\psi^I \dot\varphi^J - \varepsilon\dot\varphi^I \dot\psi^J - \psi^I \varepsilon\ddot\varphi^J - \varepsilon\dot\varphi^I \dot\varphi^K \Gamma^J_{KL}(\varphi)\psi^L + \psi^I \varepsilon\dot\psi^K \Gamma^J_{KL}(\varphi)\psi^L$$
$$+ \psi^I \dot\varphi^K \Gamma^J_{KL,M}\, \varepsilon\psi^M \psi^L - \psi^I \dot\varphi^K \Gamma^J_{KL}(\varphi)\, \varepsilon\dot\varphi^L \Big)$$

Orange terms $\quad -\varepsilon\psi^I \dot\varphi^J \frac{d}{dt}(g_{IJ})$ $\quad$ (after $\int$ by parts)

Blue terms $\quad \varepsilon\psi^M \dot\varphi^I \dot\varphi^J \left(g_{IJ,M} - g_{IL}\Gamma^L_{JM} + g_{LM}\Gamma^L_{IJ} \right) = \varepsilon\psi^M \dot\varphi^I \dot\varphi^J g_{JM,I} = \varepsilon\psi^M \dot\varphi^J \frac{d}{dt}(g_{JM})$

So these indeed cancel one another. (But should still check the green terms also vanish!)]

Classical solutions: $\mathcal{S} = \{(\varphi,\psi) : \nabla_t \dot\varphi = R(\psi,\dot\varphi)\psi,\ \nabla_t \psi = 0\} \simeq (T \oplus \Pi T) Y$
$\simeq (T^* \oplus \Pi T^*) Y$

Local coords on $\mathcal{S}$: x^I on Y, p_I on T^* fiber, $\tilde p_I$ on ΠT^* fiber

Poisson brackets $\{x^I, p_J\} = \delta^I_J$ $\quad \{p_I, p_J\} = R^{IJKL} \tilde p_K \tilde p_L$ $\quad \{\tilde p_I, \tilde p_J\} = g_{IJ}$

Hamiltonian $H = \frac{1}{2}\|p\|^2$ (generates W acting on $\mathcal{S}$)

Supercharge $Q = \langle p, \tilde p\rangle$ (odd function; generates V acting on $\mathcal{S}$)

$\{Q,Q\} = 2H$ as it should.

Canonical quantization:

First consider the case of $Y = \mathbb{R}^{2k}$ flat.
Then quantizing the linear functions would mean finding operators with

$$[x^I, p_J] = \delta^I_J \qquad [\tilde p_I, \tilde p_J] = \delta_{IJ}$$

As before, there's a unique irrep ("super version" of Stone-von Neumann Thm):

$\mathcal{H} = L^2(\mathbb{R}^{2k}) \otimes S(\mathbb{R}^{2k})$ $\qquad$ S = spin representation of $\mathcal{C}\ell(\mathbb{R}^{2k})$

$x^I \rightsquigarrow$ mult. by x^I $\qquad$ $H \rightsquigarrow \frac{1}{2}\Delta = \frac{1}{2}\sum_{I=1}^{2k} \partial_I^2 \otimes 1$

$p_J \rightsquigarrow \frac{\partial}{\partial x^J}$ $\qquad$ $Q \rightsquigarrow \not{\partial} = \sum_{I=1}^{2k} \partial_I \otimes \rho(e_I)$

$\tilde p_I \rightsquigarrow \rho(e_I)$ (spin rep)

$\not{\partial}^2 = \sum_{I,J} \partial_I \partial_J \otimes [\rho(e_I), \rho(e_J)] = \sum_I \partial_I^2 \otimes 1 = \Delta$, i.e. $Q^2 = 2H$ as it should!

If Y not flat (but still $2k$-dimensional) and obeys a top. constraint, we can globalize the above:

Def Let $P \to Y$ be the bundle of oriented orthogonal frames $(P_y = \{\text{isom. } T_y Y \to \mathbb{R}^{2k}\})$
A principal $SO(2k)$-bundle. A spin structure on Y is a $Spin(2k)$-bundle $\tilde P \to Y$ which double-covers P, in a way compatible with the double cover $Spin(2k) \to SO(2k)$.

Fact Given a spin structure $\tilde P$ on Y, there is a Hermitian vector bundle $S \to Y$ of rank 2^k, equipped with "Clifford action" $e : \Gamma(TY) \to \Gamma(\mathrm{End}\, S)$ which restricts on each fiber to an action of Clifford algebra $Cliff(T_y Y)$ on $S(T_y Y)$, and with connection ∇.

So: assume Y admits a spin structure, and fix one.

Then, our Hilbert space is $\mathcal{H} = L^2(S)$.

In fact $S = S^0 \oplus S^1$, so $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. Each is a Hilbert space.

So $\mathcal{H}$ is super/graded Hilbert space. (It happens in any theory w/ fermions.)

Dirac operator $\not{\partial} : \mathcal{H}^{\cdot} \to \mathcal{H}^{\cdot+1}$ given in ON basis by $\not{\partial} = \sum \rho\left(\frac{\partial}{\partial x^I}\right) \nabla_{\frac{\partial}{\partial x^I}}$

Spinor Laplacian $\Delta : \mathcal{H}^{\cdot} \to \mathcal{H}^{\cdot}$ given by $\Delta = \not{\partial}^2$

Our proposed quantiz^n: $f(x) \rightsquigarrow$ mult. by $f(x)$, $v \in \Gamma(TY) \rightsquigarrow \nabla_v$; $Q \rightsquigarrow \not{\partial}$, $H \rightsquigarrow \Delta$

Computing Z_{S^1} in this theory will give $Z_{S^1_\beta} = \mathrm{STr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}^0}\, e^{-\beta H} - \mathrm{Tr}_{\mathcal{H}^1}\, e^{-\beta H}$

What's that?

Let's consider rep theory of the superalgebra $\mathcal{A}$ of dim $1|1$, gen by Q odd, H even, $Q^2 = H$

Say V is a (graded) irrep of $\mathcal{A}$ which is unitary: $H^* = H$, $Q^* = Q$ acting on V

H is central in $\mathcal{A}$, so it should act as a scalar $E \in \mathbb{R}$: $H = E$ on V

Take any $|\psi\rangle \in V$. Then $\| Q|\psi\rangle \|^2 = \langle \psi | Q^2 | \psi \rangle = E \, \| |\psi\rangle \|^2$

So $Q|\psi\rangle = 0 \iff E = 0$.

In any case $Q^2|\psi\rangle = E|\psi\rangle$. So, classif^n of irreps V of $\mathcal{A}$:

- if $E \neq 0$, $\dim(V) = 1|1$ $\;(|\psi\rangle, Q|\psi\rangle)$ and $\mathrm{STr}_V\, e^{-\beta H} = e^{-\beta E} - e^{-\beta E} = 0$
- if $E = 0$, $\dim(V) = 1|0$ and $\mathrm{STr}_V\, e^{-\beta H} = 1$
  $\dim(V) = 0|1$ and $\mathrm{STr}_V\, e^{-\beta H} = -1$

So, $\mathrm{STr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{STr}_{\mathrm{Ker}\, H}\, 1 = \mathrm{SDim}(\mathrm{Ker}\, H) = \mathrm{SDim}(\mathrm{Ker}\, Q)$

This is also called the index of $Q^0 : \mathcal{H}^0 \to \mathcal{H}^1$: $\mathrm{ind}\, Q^0 = \dim \mathrm{Ker}\, Q^0 - \dim \mathrm{Coker}\, Q^0$

and $\mathrm{Coker}\, Q^0 \simeq \mathrm{Ker}\, Q^1$ [exercise]

This perspective makes deformation invariance clear: when we vary the metric in Y, the rep^n $\mathcal{H}$ of $\mathcal{A}$ varies continuously, but elements can only leave the $E=0$ locus in pairs — which don't contribute to the supertrace.

We might expect to compute $Z_{S^1_\beta}$ by localization to fixed pts of this vector field:

$$\{(\varphi,\psi) : \varphi \text{ constant}, \psi = 0\} \subset \mathcal{C}.$$

This is a trickier situation than we've met before — fixed pts are not isolated.

To study it systematically: $S = \int dt\, \frac{1}{2}\|\dot\varphi\|^2 + \frac{1}{2}\langle \psi, D_t \psi\rangle$ [rescale $t = \beta t'$, $\psi = \frac{1}{\sqrt{\beta}}\psi'$]

$$= \frac{1}{\beta}\int dt\, \frac{1}{2}\|\dot\varphi\|^2 + \frac{1}{2}\langle \psi, D_t\psi\rangle$$

Now, use the fact that $Z_{S^1_\beta}$ is indep of β. So we can take the limit $\beta \to 0$.

Localization to absolute minima of S.

If $\dim Y = n$, we have an $n|n$-dimensional locus for which $S = 0$;

$$\underset{\Pi TY}{\underset{\wr\wr}{\mathcal{M}}} = \{(\varphi,\psi) : \varphi, \psi \text{ constant}\} \subset \mathcal{C}.$$

Expanding around a general point of $\mathcal{M}$, to quadratic order: $\varphi = \varphi_0 + a(t)$, $\psi = \psi_0 + \eta(t)$

To this approx, $\mathcal{C}$ looks linear: $a \in \Gamma(\varphi^* TY)$, $\eta \in \Gamma(\varphi^* \Pi TY)$, w/ $\int dt\, a(t) = 0$, $\int d\eta\, \eta(t) = 0$.

$$S \approx \frac{1}{2\beta}\int dt\, \|\dot a\|^2 + \langle \eta, \dot\eta\rangle - \frac{1}{2}\psi_0^I \psi_0^J R_{IJKL}\, a^K \dot a^L$$

To compute Z, we'll have to do Gaussian integrals over a, η.

$\int \mathcal{D}\eta\, \exp\left(-\frac{1}{2\beta}\int dt\, \langle \eta, \dot\eta\rangle\right)$ just gives some constant, indep of φ_0, ψ_0 (after all, it should be an inv't of the Euclidean vector space $T_{\varphi_0}Y$, but there are none...)

$$\int \mathcal{D}a\, \exp\left(-\frac{1}{2\beta}\int dt\, \|\dot a\|^2 + \frac{1}{2}\psi_0^I\psi_0^J R_{IJKL}\, a^K \dot a^L\right)$$

$$= \int \mathcal{D}a\, \exp\left(-\frac{1}{2\beta}\int dt\, \langle D_B(a), a\rangle\right) \quad \text{where} \quad D_B = -\frac{d^2}{dt^2} - \frac{1}{2}\psi_0^I\psi_0^J R_{IJ}\frac{d}{dt}$$

[$R_{IJ} \in so(T)$ curvature operator]

Doing this Gaussian integral will produce $\dfrac{1}{\sqrt{\det D_B}}$.

So, up to a constant, we should get $Z_{S^1} = \displaystyle\int_{\Pi TM} d\varphi_0\, d\psi_0\, \frac{1}{\sqrt{\det D_B(\psi_0,\varphi_0)}}$

Computing this determinant: an auxiliary step —

Fix any element $g \in so(V)$ and consider $D_g = -\frac{d^2}{dt^2} - \frac{1}{2} g \frac{d}{dt}$

acting on maps $a : S^1 \to V$ w/ $\int dt\, a(t) = 0$

Diagonalize $g \Rightarrow$ reduce to case $\dim(V) = 2$, $g = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$ eigenvalues $\pm i x$

Then diagonalize D_g: $\det D_g = \prod_{k \neq 0} \prod_{\pm} \left((2\pi k)^2 \pm i (2\pi k) \frac{x}{2} \right)$

As with the harmonic oscillator, this product is divergent. It should be regulated as we did there. Get

$$\frac{1}{\sqrt{\det D_g}} = C \cdot \frac{\frac{x}{2}}{\sinh \frac{x}{2}}$$

When $\dim(V) = 2n$, similarly get $\frac{1}{\sqrt{\det D_g}} = C^n \cdot \prod_{i=1}^{n} \frac{\frac{x_i}{2}}{\sinh \frac{x_i}{2}}$

This product has another name: it is called $\hat{A}(g)$. [Some function of the entries of g.]

We want to compute $\frac{1}{\sqrt{\det D_B}}$ with $B = \frac{1}{2} \psi_0^I \psi_0^J R_{IJ}$: just apply this same function to the entries of B (each of which is a f^n on TY) to define $\hat{A}(B)$.

Then finally,

$$\operatorname{ind}(Q^0) = C^n \int_{\Pi TY} \hat{A}(\psi_0^I \psi_0^J R_{IJ})$$

$$= C^n \int_{TY} \hat{A}(R)$$

The Atiyah-Singer index formula for Dirac operators!