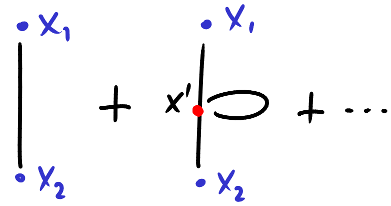
Interacting scalar in $d=4$

Scalar field with $S = \int_X \frac{1}{2}\|d\varphi\|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \quad \mathcal{C} = \{\varphi : X \to \mathbb{R}\}$

$\langle \varphi(x_1)\varphi(x_2)\rangle =$ 

$$= D(x_1,x_2) + \frac{\lambda}{2}\int dx'\, D(x_1,x')D(x_2,x')D(x',x') + \cdots$$

Here $D(\cdot,\cdot)$ is the Green's function for the Laplacian: $(\Delta_{x_1}+m^2)\,D(x_1,x_2) = \delta(x_1,x_2)$

If $X=\mathbb{R}^4$ flat and $m=0$, $D(x,y) = \dfrac{1}{\|x-y\|^2}$ $\quad$ (cf. $X=\mathbb{R}^1$, $D(x,y)=\frac{1}{2}|x-y|$)

so $D(x',x')$ is infinite! (even if $m \neq 0$)

How to interpret this ∞?

First observe: it "comes from the high energy part of the theory"! e.g. on flat $\mathbb{R}^4$,

$$D(x,y) = \int_{(\mathbb{R}^4)^*} dp\, \frac{e^{ip\cdot(x-y)}}{\|p\|^2+m^2} \qquad \text{so } D(x,x) = \int dp\, \frac{1}{\|p\|^2+m^2}$$

; divergence comes from the region of large $\|p\|$

A way to remove the problem: view our action S not as fundamental but as an effective action.

So, path integrals run only over $\mathcal{C}_\Lambda = \left\{ \begin{array}{l} \varphi : \mathbb{R}^4 \to \mathbb{R} \\ \hat{\varphi}(p) = 0 \text{ for } \|p\| > \Lambda \end{array} \right\}$

In the perturbative analysis of path integrals over $\mathcal{C}_\Lambda$,
$D(x,y)$ gets replaced by a cutoff version:

$$D_\Lambda(x,y) = \int_{\|p\|<\Lambda} dp\, \frac{1}{\|p\|^2+m^2}\, e^{ip\cdot(x-y)}$$

This renders answers finite (at least if $m \neq 0$).

But they depend on Λ. If $\Lambda \gg E$ with E the energy scale we want to study, this will cause problems: for example,

$$\langle \phi(x_1)\phi(x_2) \rangle = D(x_1,x_2) + \left[(\#)\cdot\lambda\cdot\Lambda^2 \|x\|^2 + \cdots \right] + \cdots$$

So even if the coupling λ is small, this perturbation expansion may not be well behaved.

That might be OK. After all, λ isn't directly observable anyway.

We should try instead to formulate things in terms of even more effective action $S_{eff}(E)$, where we've integrated out modes between Λ and E. What kind of terms will occur?

General expectation: $S_{eff}(E)$ is non-local. Expanding yields an infinite series of interactions. That seems terrible: how could you ever use such a theory for any practical purpose?

To investigate more closely: convenient to integrate out in way that lowers Λ "continuously".

This can be done precisely ("exact RGE"): writing the interaction part S_{int}, cutoff function ρ,

Polchinski "Renormalization and effective Lagrangians" (also Costello)

$$\Lambda \frac{\partial S_{eff}}{\partial \Lambda} = -\int d^4p \, \frac{(2\pi)^4}{2} \, \frac{1}{\|p\|^2 + m^2} \left[\frac{\partial}{\partial \Lambda} \rho\left(p^2/\Lambda^2\right) \right] \left[\frac{\delta S_{int}}{\delta \hat{\phi}(-p)} \frac{\delta S_{int}}{\delta \hat{\phi}(p)} + \frac{\delta^2 S_{int}}{\delta \hat{\phi}(p) \delta \hat{\phi}(-p)} \right]$$

This defines a flow in the ∞-dim space of possible Lagrangians.

Key idea: as $\Lambda \to 0$ this flow is driven to a 3-dimensional subspace!

i.e. if we start from a very high scale Λ and flow down to Λ_0, there is only a 3-dimensional space of possible effective theories that we can get, up to corrections which are suppressed by powers of Λ_0/Λ.

This is why QFT has some power: after measuring finitely many parameters, everything else is determined (& in a computationally effective way...)

To understand why this should be, we should think a little about scaling.

There's an action ρ_ε of the group $\mathbb{R}^\times$ on $\mathcal{C}$ by $\phi(x) \to \frac{1}{\varepsilon}\phi(\varepsilon x)$

This action was chosen so that it leaves the kinetic term $\frac{1}{2}\int \|d\phi\|^2$ invariant.

But it transforms the other terms.

$$\rho_\varepsilon^*\left(\int \phi^n\right) = \varepsilon^{4-n}\int \phi^n$$

$$\rho_\varepsilon^*\left(\int \|d\phi\|^m \phi^n\right) = \varepsilon^{4-(2m+n)}\int \|d\phi\|^m \phi^n$$

Define the scaling dimension of the coupling: $\dim \phi^n = n$

$\dim \|d\phi\|^m \phi^n = 2m+n$

$$\left[\frac{1}{\varepsilon^2}\langle \phi(0)\,\phi(\varepsilon x)\rangle \text{ computed with action } S\right] = \left[\langle \phi(0)\,\phi(x)\rangle \text{ with } \rho_\varepsilon^*(S)\right]$$

$\Longrightarrow$ at least naively (w/o thinking about role of cutoffs), we'd say that for large ε (long distance) the effects of the terms with $\dim > 4$ become small. Call these terms "irrelevant".

More gen,

$\dim > 4$ irrelevant

$\dim = 4$ marginal

$\dim < 4$ relevant

There are 3 marginal or relevant terms possible which are invariant under $\phi \mapsto -\phi$ and invariant under the Poincare group: $\phi^2, \phi^4, \|d\phi\|^2$.

This is really just dimensional analysis so far.

Another way to express the same idea: the terms in S will always contribute to correl. func. with some powers of E to make them dimensionless. So we should measure them that way: $$S_{int}(\Lambda) = g_4\phi^4 + g_6\Lambda^2\phi^6 + g_8\Lambda^4\phi^8 + \cdots$$

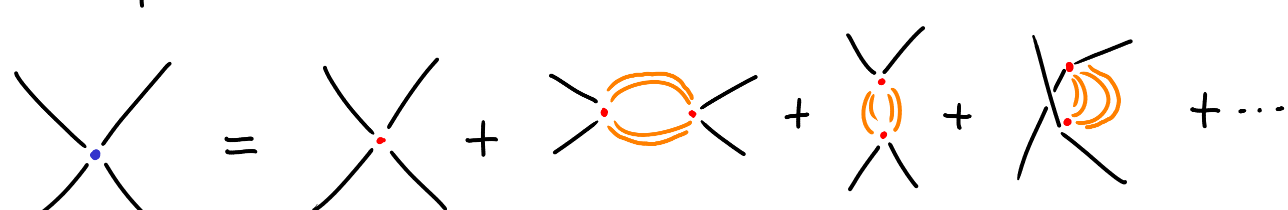
Then, the <u>classical</u> running would be $\Lambda\frac{dg_4}{d\Lambda} = 0,\ \Lambda\frac{dg_6}{d\Lambda} = 2g_6, \ldots$

The <u>exact</u> running is
$$\Lambda\frac{dg_4}{d\Lambda} = \tilde\beta_4(g_4, g_6, \ldots)$$
$$\Lambda\frac{dg_6}{d\Lambda} = 2g_6 + \tilde\beta_6(g_4, g_6, \ldots)$$
$$\vdots$$
with $\tilde\beta_k$ computable in perturbation theory.

Studying toy examples we see that the initial value of g_6 (and all other irrelevant couplings) gets <u>damped out</u> exponentially fast as $\log\Lambda$ decreases.
It's <u>proven</u> that this really happens in this theory (at least in perturbation) [Polchinski]

Even <u>after</u> the flow converges onto the finite-dim. subspace coord. by marginal and relevant couplings, at $\Lambda \ll \Lambda_0$, we still have a nontrivial flow of couplings with Λ. Coordinatizing by $S = \frac{1}{2}Z_2\|d\phi\|^2 + \frac{1}{2}m^2\phi^2 + \frac{g_4}{4!}\phi^4 + \cdots$,
$$\Lambda\frac{dg_4}{d\Lambda} = \beta_4\left(g_4, \frac{m^2}{\Lambda^2}, Z_2\right)$$
β_k can be computed in perturbation theory, e.g. in the limit $m\to 0$ and $\Lambda_0 \to \infty$,
$$\beta_4 = \frac{3g_4^2}{16\pi^2} + \mathcal{O}(g_4^3)$$

<u>Morally</u>, this comes from the relation: (using $=$ for lines integrated only over scales between Λ and Λ')



$$g_4(\Lambda') = g_4(\Lambda) + 3\cdot\frac{1}{2}g_4^2\int_{\Lambda'<\|p\|<\Lambda}\frac{dp}{(2\pi)^4 p^4} = g_4(\Lambda) + \frac{3g_4(\Lambda)^2}{16\pi^2}(\log\Lambda - \log\Lambda') + \cdots$$

For a systematic treatment see [Hughes-Lin]

This says that the effective theory (always at scales $\Lambda \ll \Lambda_0$) becomes <u>more strongly coupled</u> when we increase Λ.

$\Longrightarrow$ perturbation theory becomes less and less effective as we go to higher energies.

For any finite Λ_0, the theory <u>exists</u>, but it's hard to understand at energies where $g_4 \gtrsim 1$.

We don't really understand whether it is possible to remove the cutoff <u>completely</u>,
i.e. to take $\Lambda_0 \to \infty$ while holding $g_4(\Lambda)$ fixed.

Usually summarized by saying this theory exists "<u>only</u> as an effective theory"

This eff. theory is <u>IR free</u>: effect of interactions disappears at low enough energies.