Abelian gauge theory in 4 dimensions

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X compact oriented

$$\mathcal{C} = \left\{ \begin{array}{l} P \text{ principal } U(1) \text{ bundle over } X \\ \nabla \text{ connection in } P \end{array} \right\}$$

Let $F =$ curvature of ∇ $\qquad F \in \Omega^2(X)$

$$S = \frac{1}{2g^2}\int_X F \wedge *F + \frac{i\theta}{4\pi^2}\int_X F \wedge F$$

$\mathbb{R} \ni g =$ coupling constant

$\mathbb{R} \ni \theta =$ theta angle

Classical equations: an infinitesimal deformation of ∇ is $\nabla \rightsquigarrow \nabla + \delta\alpha$, $\delta\alpha \in \Omega^1(X)$. Then

$$\delta S = \frac{1}{2g^2}\int_X d(\delta\alpha) \wedge *F + F \wedge * d(\delta\alpha)$$

$$= \frac{1}{g^2}\int_X d(\delta\alpha) \wedge *F$$

$$= \frac{1}{g^2}\int_X \delta\alpha \wedge d*F$$

So the classical eq. say $d*F = 0$.

We also have Bianchi identity $dF = 0$ (since locally $F = dA$).

Relation to Maxwell's equations as learned in HS: take $X = \mathbb{R}^4$, expand

$$F = \begin{array}{c} \begin{array}{cccc} t & x^1 & x^2 & x^3 \end{array} \\ \begin{bmatrix} 0 & iE_1 & iE_2 & iE_3 \\ -iE_1 & 0 & B_3 & -B_2 \\ -iE_2 & -B_3 & 0 & B_1 \\ -iE_3 & B_2 & -B_1 & 0 \end{bmatrix} \end{array},$$

then

$$\begin{array}{c} dF = 0 \\ d*F = 0 \end{array} \longleftrightarrow \begin{array}{l} \operatorname{div} \vec{E} = 0 \\ \operatorname{div} \vec{B} = 0 \\ i \operatorname{curl} \vec{E} = -\dfrac{d\vec{B}}{dt} \\ \operatorname{curl} \vec{B} = i\dfrac{d\vec{E}}{dt} \end{array}$$

It will be convenient to rewrite this using $\tau = \frac{\theta}{\pi} + i\frac{2\pi}{g^2}$

$F_\pm = \frac{1}{2}(F \pm \star F)$

Then $$S = \frac{i\bar{\tau}}{4\pi}\int_X \|F_+\|^2 - \frac{i\tau}{4\pi}\int_X \|F_-\|^2$$

In computing path integrals in this theory we meet a new problem:

There are lots of equivalences between different pairs (P, ∇). ("gauge symmetry")

We want to somehow sum over "all (P,∇) modulo equivalence."

A first attempt: fix one bundle P_i in each topological class.

Then $\mathcal{C} = \bigsqcup_i \mathcal{C}_i$, and we could try integrating over each $\mathcal{C}_i$ separately.

But this still leaves the group of autoequivalences of a given bundle P_i.

This group is just $\mathcal{G} = Map(X, S^1)$.

[Concretely, a given $\chi \in Map(X, \mathbb{R}/\mathbb{Z})$ acts by $\nabla \mapsto \nabla^\chi = \nabla + d\chi$; this clearly leaves F invariant, so $S(\nabla) = S(\nabla^\chi)$]

$Vol(\mathcal{G}) = \infty$, so if we don't deal with this somehow we will get $Z = \infty$.

Formally speaking, we want to integrate not over $\mathcal{C}_i$ but over $\mathcal{C}_i / \mathcal{G}$.

We'll see this arise in concrete computations...

If we are able to define the theory, we expect it would have a symmetry

$\tau \to \tau + 2$, (following from the fact that $\int_X F \wedge F \in 4\pi^2 \mathbb{Z}$

so that e^{-S} is unchanged by this shift.)

Duality

Maxwell eq. have an evident symmetry: $F \longleftrightarrow *F$
(Exchanges the electric and magnetic fields.)

This is a shadow of a duality of QFT's which also takes $\tau \longrightarrow -\frac{1}{\tau}$.

To understand this, again consider a "big" theory

$$\mathcal{C}_{big} = \begin{cases} (P,\nabla) & U(1) \text{ bundle w/conn. over } X \\ (P_D,\nabla_D) & \text{" " " " "} \\ G \in \Omega^2(X) & \end{cases}$$

We'll design it with a symmetry that takes $\left[\begin{array}{c} (P,\nabla) \longrightarrow (P,\nabla) \otimes (P',\nabla') \\ G \longrightarrow G + F_{\nabla'} \end{array}\right]$

$\mathcal{F} = F_\nabla - G$ is invariant under this symmetry.

$$S_{big} = \frac{i\bar{\tau}}{4\pi}\int_X \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi}\int_X \|\mathcal{F}_-\|^2 - \frac{i}{2\pi}\int F_D \wedge G$$

Two options:

- Integrate out (P_D, ∇_D): – for each fixed topology of P_D, integral over ∇_D imposes $dG = 0$
  – sum over topologies of P_D says $\left[\frac{G}{2\pi}\right] \in \Omega^2(X,\mathbb{Z})$

  So G is curvature of some connection. Use the extra symmetry to set $G=0$. Then reduced to

$$S = \frac{i\bar{\tau}}{4\pi}\int_X \|F_+\|^2 - \frac{i\tau}{4\pi}\int_X \|F_-\|^2$$

- Use the symmetry to set (P,∇) to be trivial. Then complete the square $G' = G - \frac{1}{\bar{\tau}}(F_D)_+ + \frac{1}{\tau}(F_D)_-$ and $\int$ over G!

Get $$S = -\frac{i}{4\pi\bar{\tau}}\int_X \|(F_D)_+\|^2 + \frac{i}{4\pi\tau}\int_X \|(F_D)_-\|^2$$

So, have duality between 2 different descrip. of the theory, exchanging $\tau \longleftrightarrow -\frac{1}{\tau}$!

Combining this with $\tau \to \tau+2$ get a big subgp of $SL(2,\mathbb{Z})$.

(If the intersection form on X is even, then we even have a symmetry under $\tau \to \tau+1$, so in this case get the full $SL(2,\mathbb{Z})$.)

Naively Z would be invariant under this. [But we haven't been very careful about τ dependence of the infinite determinants coming from Gaussian $\int$'s. If more careful, find Z is modular, with hol/antihol wts $\frac{1}{4}(\chi-\sigma, \chi+\sigma)$.]

$F \longleftrightarrow *F$ under this duality.

i.e. $\langle F(x_1)F(x_2)\rangle_\tau = \langle *F(x_1) *F(x_2)\rangle_{-\frac{1}{\tau}}$ etc.

Remarks: 1) Why use $U(1)$ conn, not just 1-forms? b/c we will try to get this as effective theory from a higher-energy theory w/ compact G. (Ppl try to do this in "real world" too!)

2) The $SL(2,\mathbb{Z})$ duality can be viewed as coming from compactif^n of a 6-dimensional field theory on a 2-torus.

Coupling to matter:

Classically we know that an electric charge/current $j = \begin{bmatrix} i\rho \\ j_1 \\ j_2 \\ j_3 \end{bmatrix} \in \Omega^1(\mathbb{R}^4)$

[with $d*j = 0$] modifies Maxwell eq to
$$dF = 0$$
$$d*F = ig^2 *j$$

To write a QFT of e/m fields interacting w/ this fixed background, we would like to add a term to S, looking locally like "$i\int A \wedge *j$"

If P is trivial we can literally write $\nabla = d + A$ and add this term.

It has the right classical eq. and evidently does not spoil the symmetry under $A \to A + d\chi$ when $\chi: X \to \mathbb{R}$. For $\chi: X \to S^1$ this symmetry is subtler: requires that $*j \in H^3(X,\mathbb{Z})$. (Quantization of charge)

[If P is not trivial then it is subtle to write this term. In the special case where $*j$ is exact we can do it: say $*j = d\psi$.

Trivialize on patches U_i,
take partition of 1, write $\sum_i \rho_i \int_X A_i \wedge *j + \sum_i \int_X d\rho_i \wedge A_i \wedge \psi$ where $*j = d\psi$

This is gauge invariant and has the correct eq. of motion (using $\sum_i d\rho_i = 0$)

If $*j$ not exact then we can't write such a coupling — indeed the path $\int$ should really be zero in this case...]

Taking $*j$ to be a δ-function supported on a loop $\gamma \subset X$ we can also think of this coupling as inserting an operator ("Wilson line")

$$\mathrm{Hol}_{V_k,\gamma}(\nabla) \quad \left(= e^{ik\oint_\gamma A} = e^{ik\int_X *j \wedge A}\right)$$

If γ is homologically nontrivial then $\langle \mathcal{O}_\gamma \rangle = 0$, b/c of the symmetry $(P,\nabla) \to (P,\nabla) \otimes (\mathcal{L},\nabla')$ with ∇' flat but $\mathrm{Hol}_{\nabla'}(\gamma) \neq 1$

How about coupling to a dynamical field?
Let V_k be a rep^n of $U(1)$. (irreducible: so $\dim_{\mathbb{C}} V_k = 1$, $e^{i\vartheta} \mapsto e^{ik\vartheta}$ for some $k \in \mathbb{Z}$)
Let's introduce a new field φ which is a section of the associated bundle

$E_k = P \times_{U(1)} V_k$ (complex line bundle over X). So $\mathcal{C} = \left\{ \begin{matrix} (P,\nabla) & U(1)\text{-conn.} \\ \varphi & \text{section of } E_k \end{matrix} \right\}$

Conn. in P induces a conn D in E_k.

In local triv, a section of E_k means a map $\varphi: X \to \mathbb{C}$, and the connection acts by e.g.

$$D\varphi = (d\varphi + ik \cdot A\varphi) \in \Omega^1(X, \mathbb{C})$$

Now we write $S = S_{gauge} + \frac{1}{2}\int_X \|D\varphi\|^2 + \frac{1}{2}\int_X m^2\|\varphi\|^2$

$$= S_{gauge} + \frac{1}{2}\int_X \|d\varphi + ik \cdot A\varphi\|^2 + m^2\|\varphi\|^2$$

locally,

$$= S_{gauge} + \frac{1}{2}\int_X \eta^{ij}\left[\partial_i\varphi\partial_j\varphi + 2ik\partial_i\bar{\varphi}A_j\varphi - 2ik\partial_i\varphi A_j\bar{\varphi} + k^2A_iA_j|\varphi|^2 + m^2|\varphi|^2\right.$$

Eq. of motion now say: $d{*}F = g^2 ik\underbrace{(\bar{\varphi}D\varphi - \varphi D\bar{\varphi})}_{\text{call this } *j}$ (from var^n of A)

$D{*}D\varphi = m^2\varphi$ (from var^n of φ)

[Aside: Recall Noether's Thm in 1 dimension: a conserved charge for every symmetry (that acts locally on the fields). Essentially same argument applies in 4 dimensions and produces a conserved current. Here, the symmetry $\varphi \to e^{i\alpha}\varphi$ is responsible for $d(*j) = 0$]

This describes physics of the EM field coupled to massive, electrically charged particles.

[Nobody knows how to do s.t. analogous for magnetically charged particles.
If we only want magnetic ones, we can use duality to relate them to electric ones.
But if we want both electric and magnetic ones, we would be in a jam...]

For perturbation theory: first note that

$$0 \le \frac{1}{2}\int \|F \pm {*}F\|^2 = \frac{1}{2}\int (F \pm {*}F) \wedge {*}(F \pm {*}F)$$

$$= \int F \wedge F \pm \int F \wedge {*}F \quad \text{i.e. } \int F \wedge {*}F \gtrless \pm \int F \wedge F$$

$$\text{i.e. } \int F \wedge {*}F \ge \left|\int F \wedge F\right|$$

This means that the path integral sectors with $\int F \wedge F = 4\pi^2 k$

have $S \geq 2\pi^2 \frac{|k|}{g^2}$

and so come weighed by a factor $e^{-\frac{2\pi^2 |k|}{g^2}}$

$\Longrightarrow$ their effects are <u>invisible</u> in a power series expansion around $g=0$!
"Nonperturbative."

To see why g is a <u>coupling</u>, look at the sector where P is trivial, then <u>rescale</u> $A \to gA$: then S becomes

$$\frac{1}{2}\int_x \eta^{ik}\eta^{jl}(\partial_i A_j - \partial_j A_i)(\partial_k A_l - \partial_l A_k) + \frac{1}{2}\eta^{ij}\left[\partial_i \varphi \partial_j \varphi + 2ik\partial_i \bar{\varphi}\, g A_j \varphi - 2ik \partial_i \varphi\, g A_j \bar{\varphi} + g^2 k^2 A_i A_j |\varphi|^2\right] + m^2 |\varphi|^2$$

The interacting theory (coupled to matter) has a running coupling: if we introduce matter fields in representations V_{k_i} $(i=1,\dots,n)$ then

$$\Lambda \frac{dg}{d\Lambda} = \frac{1}{48\pi^2} g^3 \sum_{i=1}^{n} k_i^2 + O(g^5)$$

Qualitatively similar to ϕ^4 theory: g is <u>larger</u> at higher energies!

So, the pure abelian gauge theory coupled to matter is "only" an effective theory.
