Coupling to fermions

Recall the spin representation of $Cliff(\mathbb{R}^4)$, $S \simeq S^+ \oplus S^-$ (with $Spin(\mathbb{R}^4) \circlearrowleft S^+$, $Spin(\mathbb{R}^4) \circlearrowleft S^-$).

$Spin(\mathbb{R}^4) \simeq SU(2)_+ \times SU(2)_-$ and in fact S^+ is fund^l rep. of $SU(2)_+$
S^- is " " " $SU(2)_-$

Now if X is a spin 4-manifold [w/ principal $Spin(4)$ bundle B] then we get assoc. spin bundles $S_B^\pm$. Dirac operator $\not{\partial} : \Gamma(S_B^\pm) \to \Gamma(S_B^\mp)$.

Now we can define a new theory by
$$\mathcal{E} = \begin{cases} \psi^+ \in \Gamma(\Pi S_B^+) \\ \psi^- \in \Gamma(\Pi S_B^-) \end{cases}$$

$$S = \frac{1}{2}\int_X \langle \psi^+, \not{\partial} \psi^- \rangle$$
$\langle , \rangle$ = Hermitian inner product on S^+

That's free (quadratic).

Can also couple to gauge fields:
$$\mathcal{E} = \begin{cases} (P, \nabla) \\ \psi^+ \in \Pi\Gamma(S_B^+ \otimes V_P) \\ \psi^- \in \Pi\Gamma(S_B^- \otimes V_P) \end{cases}$$

$$S = \frac{1}{2}\int_X \langle \psi^+, \not{D} \psi^- \rangle$$

$\not{D}$ = covariant Dirac op:
$\sum_i \rho(e_i) D_{e_i}$ (e_i orthonormal basis for TX)

[Our rules for path integration over these fermions will be by analytic continuation from signature $(3,1)$.]

It's a general principle of QFT that fermions usually "have $\frac{1}{2}$-integer spin"
i.e. they are associated with reps of $so(\mathbb{R}^n)$ that don't integrate to $SO(\mathbb{R}^n)$ but only to $Spin(\mathbb{R}^n)$. "Spin-statistics theorem" (but has loopholes...)

Roughly a consequence of unitarity.