The Seiberg-Witten Solution

Now let's return to $\mathbb{R}^4$ and ask: what is the physics of $\mathcal{N}=2$ SYM at "very low" energies?

We'll attempt to describe the physics by an action including only the massless fields.

What are they?

A tricky question: the global minimum of S is actually degenerate!

Any config with $F=0$, $D\phi=0$, $[\phi,\bar\phi]=0$ is a global minimum ("classical vacuum").

Such a ϕ can be diagonalized: $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$

So have a $\mathbb{C}/\mathbb{Z}_2$ worth of inequivalent classical vacua. $u = \mathrm{Tr}\,\phi^2 = a^2$.

Let's expand around one such vacuum. If $u \neq 0$, Higgs mechanism breaks $SU(2) \to U(1)$.

W bosons get masses $\sim |ga|$ as we've discussed.

Same also happens for off-diagonal components of $\lambda^\pm$.

Crude approximation: set all massive fields to zero. Then, we get a field theory of:

$$\mathcal{C} = \begin{cases} a: X \to \mathbb{C} \\ (P,\nabla) \;\; U(1) \text{ conn.} \\ \lambda^\pm \in \Gamma(S^\pm \otimes R) \end{cases}$$

i.e. $\mathcal{N}=2$ SYM, $G=U(1)$, with the same (g,ϑ) as we had in the original $G=SU(2)$ theory,

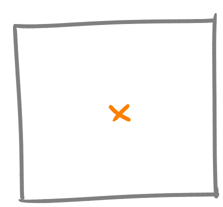
$$S = \frac{1}{g^2}\int \|da\|^2 + i\vartheta\int F\wedge F + \cdots = \int (\mathrm{Im}\,\tau)\,\|da\|^2 + \frac{i\tau}{4\pi}\int \|F_+\|^2 + \frac{i\bar\tau}{4\pi}\int \|F_-\|^2$$

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$$

Recall $u = a^2$, so can also write this as $S = \frac{1}{g^2}\int \frac{\|du\|^2}{|u|^2} + \cdots$

This part describes a σ-model into the space of vacua, with Kähler metric;

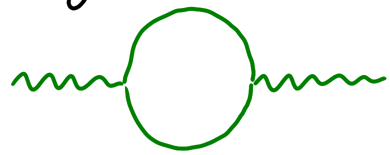
Singularity at $u=0$. Interpretation: this is the pt. where W bosons become massless.



But this isn't the exact action: we should include interactions coming from integrating out.

First: running coupling of the original $SU(2)$ theory. Let g_{eff}^2 be eff. coupling at scale a.

Perturbation theory $\Rightarrow$ $\dfrac{1}{g_{eff}^2} = \dfrac{1}{4\pi^2} \log \dfrac{|u|}{\Lambda_s}$



(Only 1-loop contribution — higher loops vanish!)

So, the effective action must have a coupling that depends on a.

And, it must be supersymmetric.

Turns out, this imposes a constraint: $\tau = \tau(a)$ should be holomorphic.

So the running of θ is determined by that of g: $\tau(a) = \frac{i}{\pi} \log \frac{u}{\Lambda_s}$ in pert. theory

Or, a slicker way: in the orig. $SU(2)$ th., anomaly implies that $\langle \mathcal{O} \rangle_{\vartheta_0} = \langle \mathcal{O}^\alpha \rangle_{\vartheta_0 + 8\alpha}$

[where $\mathcal{O}^\alpha$ is the $U(1)_R$ transform" of $\mathcal{O}$]

ie: acting with $U(1)_R$ can be compensated by shifting $\vartheta_0 \to \vartheta_0 + 8\alpha$.

This $\Rightarrow$ $\vartheta_{eff}(u) = \vartheta_{eff}(e^{4i\alpha} u) + 8\alpha$, in pert theory (proof required here! instantons destroy this relation)

Now, under monodromy $u \to e^{2\pi i} u$, $a \to e^{\pi i} a$ we have $\tau \to \tau - 2$, $a \to -a$

Looks bad: our action is multi-valued!

But recall that this is a duality symmetry: the abelian gauge theory w/ coupling τ and w/ coupling $\tau - 2$ are equivalent. (Part of the $SL(2,\mathbb{Z})$ duality group: $\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$)

So the low energy physics is subtle: we can describe it by an action in each domain of field space, in an $SL(2,\mathbb{Z})$ worth of ways, but cannot give a single action that works globally.

Now: what is the effective action?

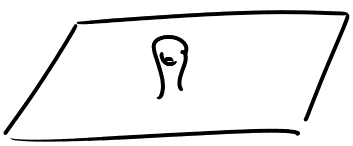
Let's formulate our problem mathematically. We seek:

- A complex curve $\mathcal{B}$ (possibly singular)
- A holomorphic function $u: \mathcal{B} \to \mathbb{C}$ $\quad [u = \langle \mathrm{Tr}\, \phi^2 \rangle]$

  such that $\exists$ an open subset $\mathcal{U} \subset \mathcal{B}$ s.t. $u: \mathcal{U} \xrightarrow{\sim} \{|u| > C\} \subset \mathbb{C}$
- A holomorphic map $\tau: \mathcal{B} \to \mathbb{H}/SL(2,\mathbb{Z})$

  such that as $|u| \to \infty$ in $\mathcal{U}$, $\tau \to \frac{i}{\pi} \log \frac{u}{\Lambda_s^2}$

The existence of u which is 1-1 on $\mathcal{U}$ implies that $\overline{\mathcal{B}} \simeq \mathbb{CP}^1$, $\mathcal{B} \simeq \mathbb{C}$.

i.e.  doesn't happen.

But then existence of τ leads to a paradox: $\mathcal{B}$ is simply connected, so

$$\begin{tikzcd} & \mathbb{H} \arrow[d] \\ \tau: \mathcal{B} \arrow[ur, dashed, "\exists"] \arrow[r] & \mathbb{H}/SL(2,\mathbb{Z}) \end{tikzcd}$$

That's inconsistent with the monodromy we found near ∞!

OK, so maybe we should allow that our effective theory breaks down at some points of $\mathcal{B}$. (Like what we saw in the classical approx.) So, let τ be defined only on the complement of some "singular locus" $D \subset \mathcal{B}$.

Could D be a single point, as we had classically?

$$\begin{tikzcd} & \mathbb{H}/\mathbb{Z} \arrow[d] \\ \text{Then } \tau: \mathcal{B}' \arrow[ur, dashed] \arrow[r] & \mathbb{H}/SL_2\mathbb{Z} \end{tikzcd}$$

So, $\mathrm{Im}\,\tau$ is a globally defined, harmonic f^n on $\mathcal{B}' \simeq \mathbb{C}^\times$. But then it can't be nonconstant and everywhere positive. (e.g. study DE obeyed by the average along S^1)

Could D be two points? Maybe...

Let's try to understand what kinds of singularities we should <u>allow</u>.

Some kinds of singularities we <u>could</u> understand: places where some particle becomes massless. Suppose we pick a duality frame where the massless particle carries only <u>electric</u> charge. Then, can write an effective Lagrangian in which this particle is represented by a <u>field</u>:

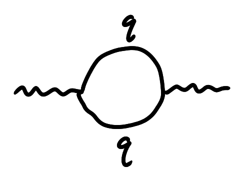
$$\mathcal{L} = \left\{ \begin{array}{l} \left.\begin{array}{l} (P,\nabla) \\ \lambda^{\pm} \\ a \in \Gamma(\mathfrak{g}_c) \end{array}\right\} \text{ as usual in } \mathcal{N}=2 \text{ gauge theory} \\ D \\ \psi^{\pm} \in \Pi\Gamma(S^{\pm} \otimes (V_c)_p) \quad \chi^{\pm} \in \Gamma(S^{\pm} \otimes (V_c)_p) \\ q \in \Gamma((V_c)_p \otimes R) \end{array} \right.$$

Action: $S = S_{gauge} + \int_X \|\nabla q\|^2 + \|(m+a)q\|^2 +$ terms involving <u>fermions</u>

This means that (expanded around a fixed a) the field q has mass $m' = |m + a\cdot c|$.

Now, suppose we <u>integrated out</u> this field.

That produces $\tau = \frac{c^2}{\pi i} \log\left(\frac{m'}{\Lambda'}\right) + \text{const.}$

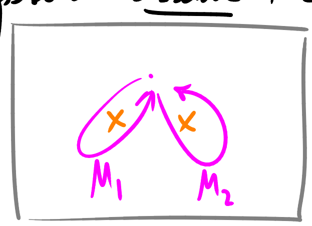


So, $\begin{cases} \text{if } c = 0 \text{ then } \tau \text{ is indep of } m' \\ \text{if } c \neq 0 \text{ then } \tau \text{ is } \underline{\text{singular}} \text{ at } m' = 0, \text{ i.e. at } a = -\frac{m}{c} \end{cases}$

The <u>monodromy</u> around this sig. pt is conj. to $\begin{pmatrix} 1 & 2c^2 \\ 0 & 1 \end{pmatrix}$

<u>So</u>: Suppose we <u>assume</u> there are exactly 2 singular points and <u>both</u> are of this type.

Put singularities, say, at $u = \pm\Lambda^2$.



$$\left.\begin{array}{l} M_1 M_2 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \\ \operatorname{Tr} M_1 = \operatorname{Tr} M_2 = 2 \end{array}\right\} \Longrightarrow M_1 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \left[\text{up to conj. by } \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right]$$

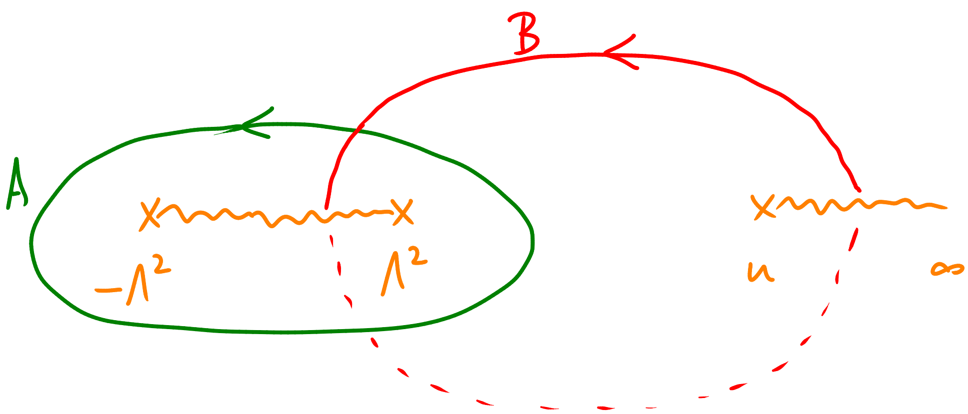
These matrices generate $\Gamma(2) = \{M \in SL_2\mathbb{Z} : M = 1 \bmod 2\}$

SW found an explicit description of τ with these monodromies:

Consider a family of elliptic curves over $\mathcal{B}'$: $\Sigma_u = \{y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)\}$

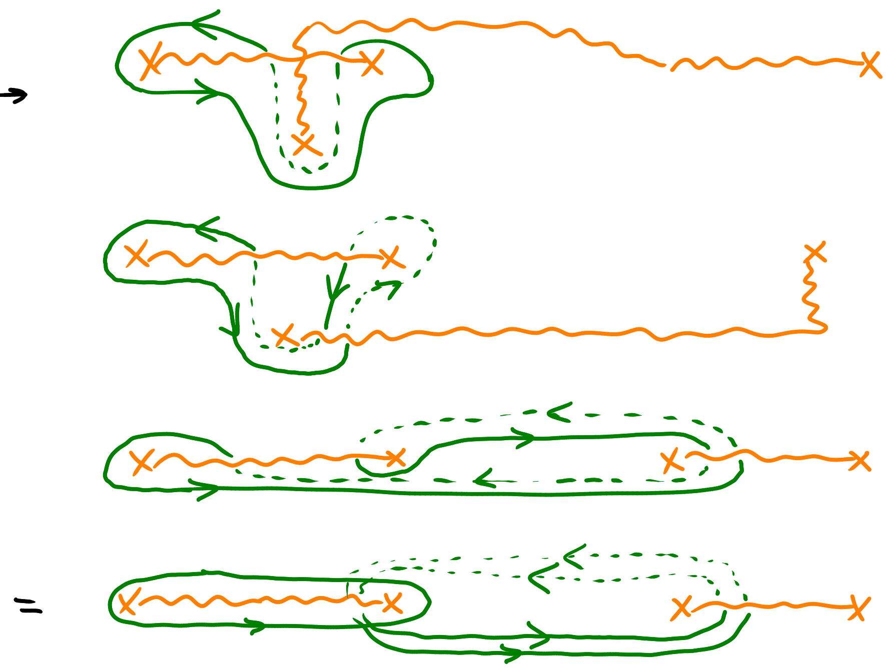
The τ for this family of elliptic curves <u>determines</u> the desired effective action!

To <u>check</u> the monodromy: note that the $SL(2,\mathbb{Z})$ action can be read off from the "Gauss-Manin connection" in $H_1(\Sigma, \mathbb{Z})$



Upon moving u around Λ^2, A and B are <u>dragged into</u> cycles A' and B':

A' obtained by $\longrightarrow$



$= A + 2B$

And similarly (but much more easily), $B' = B$.

So $M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ indeed. Similarly, check $M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

To <u>discover</u> this family of curves: note M_1, M_∞ generate $\Gamma(2)$

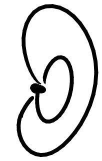
So we want a family of "elliptic curves with level 2 structure." One way to describe that: family of cubic curves w/ 4 <u>sections</u>. $(0, \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2})$

The family we wrote down has 4 manifest sections $(u, \Lambda^2, -\Lambda^2, \infty)$

and in fact it's known to be the <u>universal family</u> of ell. curves w/ level 2 structure.

<u>Remarks</u>

- The curves Σ_u at first sound like an "unphysical" gadget, only useful as an auxiliary. But, they turned out to have a more direct meaning...
- The points $u = \Lambda^2, -\Lambda^2$ are loci where Σ_u becomes <u>singular</u> (<u>node</u>)



- To check the picture, use BPS mass formula: $M = |a| = \int_\gamma \lambda$

$$\gamma \in H_1(\Sigma, \mathbb{Z})$$

$$\lambda = \frac{dx}{y} \quad \underline{\text{meromorphic}} \text{ 1-form}$$