Mass and the BPS bound

So far we used "mass" to refer to the quadratic part of the potential for some field.

But it also has a more intrinsic meaning, as follows.

Consider a field theory, say with $X = \mathbb{R}^{3,1}$ and $S = \int_X -\frac{1}{2}\|d\phi\|^2 + \frac{1}{2} m^2 \phi^2$

Then consider the Hilbert space in canonical quantization.

It should be a representation of $ISO(3,1)$.

Can understand $iso(3,1)$ action concretely. Recall that every symmetry v.f. X corresponds to a conserved quantity f_X.

In particular, the action of $\frac{\partial}{\partial x^i} \in \mathbb{R}^{3,1} \subset iso(3,1)$ has $f_{\frac{\partial}{\partial x^i}} = p^i = \int d\vec{x} \; \dot{\phi} \frac{\partial}{\partial x^i} \phi$

$$f_{\frac{\partial}{\partial t}} = H = \frac{1}{2} \int d\vec{x} \; \|d\phi\|^2 + m^2 \phi^2$$

Canonical quantization of these functions produces operators $\hat{p}^i, \hat{H}$ acting on the Hilbert space. (Ordering problem is mild here: only shift by a constant)

Recall that the theory decouples into an infinite set of harmonic oscillators:

$$\phi(\vec{x},t) = \int d\vec{p} \; \phi(\vec{p},t) \, e^{i\vec{p}\vec{x}} \qquad \vec{p} \in (\mathbb{R}^3)^*$$

$$S = \frac{1}{2} \int d\vec{p} \, dt \, (\vec{p}^2 + m^2) \, \phi(\vec{p},t)^2$$

Each of these harmonic oscillators has a creation operator $a^\dagger(\vec{p})$

Let $|0\rangle$ be the vacuum ($\otimes$ of vacuum for each h.o. separately).

How to interpret the state $|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle \in \mathcal{H}$?

One finds $\hat{p}_i |\vec{p}\rangle = p_i |\vec{p}\rangle$

$$\hat{H} |\vec{p}\rangle = \sqrt{m^2 + \|\vec{p}\|^2} \; |\vec{p}\rangle$$

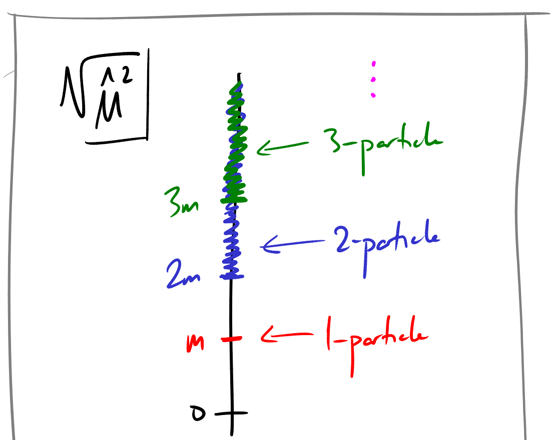
i.e. the energy-momentum of this state is $p = \begin{pmatrix} H = \sqrt{m^2 + \|\vec{p}\|^2} \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$

Obeys the relativistic energy-momentum relation: $\|p\|^2 = H^2 - \|\vec{p}\|^2 = m^2$

Acting with $SO(3,1)$ transforms $|p\rangle \to c|gp\rangle$. So $|p\rangle$ generates an ∞-dim irrep V_m of $SO(3,1)$ (of a very simple sort). $Z[U[so(3,1)]]$ contains a Casimir operator $\hat{M}^2 = \hat{H}^2 - \|\hat{\vec{p}}\|^2$ which acts as m^2 on V_m

Tentatively identify $V_{m^2} \subset \mathcal{H}$ as the space of 1-particle states.

What else is in $\mathcal{H}$? The states $a_{\vec{p}_1} a_{\vec{p}_2} |0\rangle$ (2-particle states) have $\hat{M}^2 = \|p_1 + p_2\|^2 \geq 4m^2$. Distinguished from 1-particle states by the fact that they sit in continuous spectrum of the operator $\hat{M}^2$:



In more general QFT, we (attempt to) define the "1-particle part" of $\mathcal{H}$ to be the irreps which occur discretely in the decomposition of $\mathcal{H}$ into irreps of $Spin(3,1)$.

These can be slightly more general than the V_{m^2} which we encountered above. To classify them look at the subset with $p = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$: it's an irrep of $Spin(3) \subset Spin(3,1)$, ie some V_n.

Each such V_n "induces" a corresponding irrep $V_{m,n}$.

In $\mathcal{N}=2$ SUSY QFT, we have a little more structure.

$\mathcal{H}$ is a (unitary) representation of a superalgebra $\tilde{\mathfrak{g}}$ extending $ISO(3,1)$:
$$\tilde{\mathfrak{g}}^+ = iso(3,1) \oplus \mathbb{C}$$
$$\tilde{\mathfrak{g}}^- = S^+_{\mathbb{C}} \oplus S^-_{\mathbb{C}}$$

$U(\tilde{\mathfrak{g}})$ has another generator Z, and $\tilde{\mathfrak{g}}$ has unitary irreps $\tilde{V}_{m,Z,n}$

They obey a bound $m \geq |Z|$

Moreover they divide into two types:
$m > |Z|$ "long representations"
$m = |Z|$ "short representations"

This is a fancier analogue of something we studied in SUSY QM, where we had 2-d reps for $E>0$ and 1-d reps for $E=0$. Like that case: "index" $\mathcal{I}(\mathcal{H})$ which counts the # of short reps that occur in $\mathcal{H}^1$.

(See notes on my web page for more detail about this.)

In particular, $\mathcal{I}(\mathcal{H}(u))$ is a deformation invariant of u (at least naively: this can actually be violated when the 1-particle part mixes with multiparticle part, but that violation won't affect us here. It is part of the story of wall crossing.)

In the example of interest for us, $\mathcal{H}(u) = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma(u)$ [Γ = lattice of e/m charges]

so we get a bunch of indices $\mathcal{I}(\mathcal{H}_\gamma(u))$