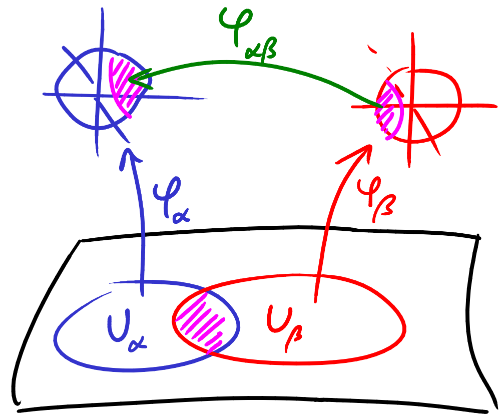
Intro

Main subject of this course: the differential geometry of complex manifolds.

"Manifold" will always mean "C^∞ manifold".

Def A holomorphic atlas on a manifold is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ where each $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ and the transition functions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic.

Two atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$ are called equivalent if all $\varphi_\alpha \circ \varphi'^{-1}_\beta$ are holomorphic.



Def A complex manifold (of complex dimension n) is a manifold (of (real) dimension $2n$) w/ an equiv. class of hol. atlas.

Ex Any orientable real surface admits a complex manifold structure.

Rk Not known whether S^6 admits a complex manifold structure. (S^2 is, all other S^k aren't)

The natural differential geometry of these objects is Kähler geometry:

Def (vague) A Kähler metric on X is a Riemannian metric for which at every point there exist Rieman normal coords which are holomorphic,

$$g = \sum_{i=1}^n dz_i \otimes d\bar{z}_i + O(\|z\|^2) \quad z_i \text{ hol.}$$

Motivations for studying Kähler manifolds:

1) AG: any smooth $\mathbb{C}$ projective variety is Kähler.
2) DG: Kähler is simplest example of "special holonomy" — many problems simplify — e.g. pb of finding Ricci-flat metrics (Einstein eq)
3) SG: Kähler metric is a natural combination of symplectic and complex structures.
4) Physics: Kähler metric is "the geometry of SUSY in $d=4$".

<u>A first goal:</u>

We will see that compact Kähler mfd carry a rich linear-algebraic structure:

e.g. cohomology <u>decomposes</u> as

$$H^n(X,\mathbb{C}) = \bigoplus_{\substack{p+q=n \\ p,q\geq 0}} H^{p,q}(X) \qquad\qquad H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$$

• A consequence: if X <u>compact</u> <u>Kähler</u> then $b_{2n+1}(X)$ is <u>even</u> $\forall n$.

This generalizes the fact that if X is a Riemann surface of genus g it has $b_1 = 2g$.

• Indeed, Kähler str. $\rightsquigarrow$ a natural <u>way of splitting</u>

$$\begin{array}{c} H^1(X,\mathbb{C}) \simeq \mathbb{C}^{2g} \\ \text{is} \\ H^{1,0}(X) \oplus H^{0,1}(X) \simeq \mathbb{C}^g \oplus \mathbb{C}^g \end{array}$$

Moreover, (in this case) this splitting contains the <u>same information</u> as the complex structure on X — so it's a linear algebraic way of encoding the $\mathbb{C}$ geometry!

<u>Other goals:</u> as listed in a short PDF on the course Web page.
(Feedback requested!)

We'll start with some purely complex-analytic stuff.
(Then, in a few lectures, we'll see how to get new insights into this stuff using a Kähler metric.)