Linear algebra of complexification

The tangent bundle of a complex mfd of dim$=n$ is a real bundle, of real dim$=2n$.
We might think it should also be a complex bundle, of dim$=n$. How to understand the relation?

Def If V is a real v.s., an almost complex structure (or complex structure!) on V is $I \in \mathrm{End}(V)$ w/ $I^2 = -\mathbb{1}$.

Prop The datum of a $\mathbb{C}$ vector space is equivalent to that of a real vector space with an almost $\mathbb{C}$ structure I.

Pf If W is a complex v.s., let V be W viewed as a real vector space. Then the op. of multiplication by i gives an almost $\mathbb{C}$ str I on V.

Conversely, a pair (V, I) canonically determines a complex v.s. W; $W = V$ as sets, with $\mathbb{C}$ acting by $(a+bi)v = av + bIv$ $[a,b \in \mathbb{R}]$ ■

Def If W is a cplx vector space corresp. to (V, I) then $\overline{W}$ is the cplx v.s. corresp. to $(V, -I)$.

But, to understand equations like $\partial/\partial z = \partial/\partial x + i\,\partial/\partial y$ we have to look at $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$, not W.

Rk For any real vector space V, $V_\mathbb{C}$ has a natural sesquilinear involution $v \mapsto \bar{v}$ ("real structure") which sends $v \otimes \lambda \mapsto v \otimes \bar{\lambda}$. Fixed locus $\simeq V$.

Def Extend I to act on $V_\mathbb{C}$ in a $\mathbb{C}$-linear way.
$V^{1,0} \subset V_\mathbb{C}$ is the $+i$-eigenspace of I. $\qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix}$
$V^{0,1} \subset V_\mathbb{C}$ is the $-i$-eigenspace of I.

Ex If $V = \mathbb{R}^2$ with $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $V^{1,0} = \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \mathbb{C}$, $V^{0,1} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \mathbb{C}$

Prop Given (V, I): a) $V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$; b) 2 summands exchanged by complex conjugation.

Pf a) $v_\mathbb{C} = \frac{1}{2}\underbrace{(v_\mathbb{C} - iIv_\mathbb{C})}_{V^{1,0}} + \frac{1}{2}\underbrace{(v_\mathbb{C} + iIv_\mathbb{C})}_{V^{0,1}}$. b) $I(\bar{v}_\mathbb{C}) = \overline{Iv_\mathbb{C}}$ since $I(v \otimes \bar{\lambda}) = Iv \otimes \bar{\lambda} = \overline{Iv \otimes \lambda}$.
If $v_\mathbb{C} \in V^{1,0}$ then $I(\bar{v}_\mathbb{C}) = \overline{Iv_\mathbb{C}} = \overline{iv_\mathbb{C}} = -i\bar{v}_\mathbb{C}$. ■

Prop Given (V, I): let W be the corresp. cplx v.s., $V^{1,0} \simeq W$, $V^{0,1} \simeq \overline{W}$.

Pf $\eta: V \to V^{1,0}$, $v_\mathbb{C} \mapsto \frac{1}{2}(v_\mathbb{C} - iIv_\mathbb{C})$. $\mathbb{R}$-linear, $\eta(Iv) = i\eta(v)$. Thus it gives $\mathbb{C}$-linear $W \to V^{1,0}$. Injective since $\eta(v) + \overline{\eta(v)} = v$. Dim count $\Rightarrow \simeq$. Similar for $\overline{W}$ ■

Def If V has $\mathbb{C}$ str I then V^* also has one, given by $(I\phi)(v) = \phi(Iv)$ $\quad \phi \in V^*$

Exterior algebra

Def If V is a vector space (in char 0), define the exterior algebra $\Lambda^\bullet(V)$ to be the tensor algebra on V, modulo the relation

$$v_1 \otimes v_2 = -v_2 \otimes v_1$$

Write the product as Λ.

So e.g. if $\dim V = 3$, choose basis $\{e_1, e_2, e_3\}$ for V, then $\Lambda^\bullet(V)$ has basis

	e_1	$e_1 \wedge e_2$	
1	e_2	$e_1 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$
	e_3	$e_2 \wedge e_3$	
$\Lambda^0(V)$	$\Lambda^1(V)$	$\Lambda^2(V)$	$\Lambda^3(V)$

$$\Lambda^\bullet(V) = \bigoplus_{k=0}^{n} \Lambda^n(V). \qquad \dim \Lambda^\bullet(V) = 2^{\dim V}.$$

Def If $\alpha \in \Lambda^k(V)$ write $|\alpha| = k$.

Prop $\Lambda^\bullet(V)$ is "graded-commutative," $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$.

Prop $V \rightsquigarrow \Lambda^\bullet(V)$ is functorial: given $f: V \to W$ get $\Lambda^\bullet(f): \Lambda^\bullet(V) \to \Lambda^\bullet(W)$, compatible with composition.

$[\Lambda^\bullet f(e_1 \wedge e_2) = f(e_1) \wedge f(e_2)]$

Prop If V is a real v.s. w/ $\mathbb{C}$ str I,

$$\Lambda^k(V_{\mathbb{C}}) = \bigoplus_{p+q=k} \Lambda^{p,q}(V) \qquad \text{with } \Lambda^{p,q}(V) \simeq \Lambda^p(V^{1,0}) \otimes \Lambda^q(V^{0,1})$$

Pf Follows from $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. e.g. use basis e_i for $V^{1,0}$, $\bar{e}_i$ for $V^{0,1}$

Then $\Lambda^{p,q}$ consists of pieces with p e_i and q $\bar{e}_i$ ∎

[e.g. $e_2 \wedge \bar{e}_1 \wedge \bar{e}_2 \in V^{1,2}$, etc.]