Forms on almost $\mathbb{C}$ manifolds

<u>Notation</u> $\Omega^k(M) = C^\infty$ sections of $\Lambda^\cdot(T^*M)$ over M.

Recall the <u>exterior derivative</u> operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$d(d\omega) = 0$$

<u>Notation</u> For M almost complex, $\Lambda^\cdot(T^*M) = \bigoplus_{p,q} (T^*)^{p,q} M$, and let $\Omega^{p,q}(M)$ be the space of C^∞ sections of $(T^*)^{p,q}M$.

Thus $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$.

<u>Def</u> Let $\pi^{p,q}$ be projection on the $\Omega^{p,q}$ summand.

Let $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ be $\pi^{p+1,q} \circ d$.

Let $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ be $\pi^{p,q+1} \circ d$.

$$\begin{tikzcd}[column sep=small]
\cdots & & \cdots & & \cdots & & \cdots \\
& \Omega^{2,0} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & & \Omega^{1,1} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & & \Omega^{0,2} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & \\
& & \Omega^{1,0} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & & \Omega^{0,1} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & & \\
& & & \Omega^{0,0} \arrow[ul, "\partial"] \arrow[ur, "\bar\partial"'] & & &
\end{tikzcd}$$

<u>Ex</u> If $\omega = z_1 z_2\, dz_2 + z_1 \bar z_2\, dz_1 \in \Omega^{1,0}(\mathbb{C}^2)$

then $d\omega = \underbrace{z_2\, dz_1 \wedge dz_2}_{\partial\omega \in \Omega^{2,0}} + \underbrace{z_1\, d\bar z_2 \wedge dz_1}_{\bar\partial\omega \in \Omega^{1,1}}$

<u>Prop</u> $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-)^{|\alpha|} \alpha \wedge \partial\beta$, $\bar\partial(\alpha\wedge\beta) = \bar\partial\alpha \wedge \beta + (-)^{|\alpha|}\alpha \wedge \bar\partial\beta$

<u>Pf</u> If $\alpha \in \Omega^{p,q}$, $\beta \in \Omega^{p',q'}$ then $d(\alpha\wedge\beta) = d\alpha \wedge \beta + (-)^{|\alpha|} \alpha \wedge d\beta$
then project on $\Omega^{p+p'+1, q+q'}$ in each term. $\blacksquare$