Holomorphic forms

On a complex mfd, $\Lambda^{p,0}(T^*X)$ all have hol. structure. (But not $\Lambda^{p,q}$!)

Notation $\Omega^p = \Lambda^{p,0}(T^*X)$ as a hol v.b.

Prop $\Gamma(\Omega^p) \simeq \{\alpha \in C^\infty(\Lambda^{p,0}T^*) \mid \bar\partial\alpha = 0\}$

Pf Coordinate calc. ∎

Hol. n-forms are naturally contour-integrated over n-dim contours. ($n = \dim_{\mathbb{C}} X$)

This follows from:

Prop $\alpha \in \Gamma(\Omega^n)$, Y an $(n+1)$-chain on X $\Rightarrow \int_{\partial Y} \alpha = 0$

Pf Stokes: $\int_{\partial Y} \alpha = \int_Y d\alpha = \int_Y (\partial + \bar\partial)\alpha = 0$ [$\partial\alpha = 0$ for degree reasons, $\bar\partial\alpha = 0$ b/c α holomorphic] ∎

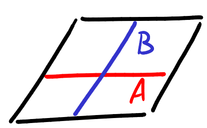
So, get a pairing between $H_n(X,\mathbb{Z})$ and $\Gamma(\Omega^n(X))$

This pairing detects a lot of information about the $\mathbb{C}$ str of X.

Ex $X = \Sigma_\tau = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$

$\Gamma(\Omega^1(X)) = \langle dz \rangle$ (because dz is global nonvanishing section $\Rightarrow$ it trivializes $\Omega(X)$)

$H_1(X,\mathbb{Z})$ is generated by A and B cycles:



$\oint_A \lambda\, dz = \lambda \qquad \oint_B \lambda\, dz = \lambda\tau$

$\Rightarrow$ for any $\omega \in \Gamma(\Omega^1(X))$, $\tau = \dfrac{\oint_B \omega}{\oint_A \omega}$

Change of basis transforms τ by $SL_2\mathbb{Z}$.