Integrability

Prop Suppose X is an almost $\mathbb{C}$ manifold.
TFAE: a) almost $\mathbb{C}$ str on X is induced by a $\mathbb{C}$ str
b) $d = \partial + \bar{\partial}$
c) $\bar{\partial}^2 = 0$, $\partial^2 = 0$, $\partial\bar{\partial} = -\bar{\partial}\partial$
d) $\bar{\partial}^2 = 0$
e) $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$

Pf a) $\Rightarrow$ b) use local coords to compute $d(f\, dz_{i_1} \wedge \cdots \wedge dz_{i_n} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_n})$
b) $\Rightarrow$ a) Newlander-Nirenberg thm (hard, we won't prove it now)
b) $\Rightarrow$ c) $d^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2 = 0$ means all 3 pieces vanish
c) $\Rightarrow$ d) trivial
e) $\Rightarrow$ b) $\alpha \in \Omega^1 \Rightarrow d\alpha(v,w) = v(\alpha(w)) - w(\alpha(v)) - \alpha([v,w])$. If $v,w \in T^{0,1}$ and $\alpha \in \Omega^{1,0}$ this gives $d\alpha(v,w) = 0$, so $d\alpha$ has no $(0,2)$ part. Similarly, if $\alpha \in \Omega^{0,1}$, $d\alpha$ has no $(2,0)$ part. Thus $d = \partial + \bar{\partial}$ when acting on Ω^1. Finally, Leibniz rule $\Rightarrow d = \partial + \bar{\partial}$ on Ω^k.
d) $\Rightarrow$ e) exercise ∎

Cor An almost $\mathbb{C}$ str on a real surface is always integrable.

Analogous situation for vector bundles:

Notation Given C^∞ cplx v.b. E over almost $\mathbb{C}$ mfd X,
$\Omega^{p,q}(E) = C^\infty$ sections of $\Lambda^{p,q} T^* X \otimes E$

Def Given hol v.b. E over $\mathbb{C}$ mfd X, define $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ by $\bar\partial_E(\alpha \otimes s) = \bar\partial\alpha \otimes s$ for s a local hol. section of E, $\alpha \in \Omega^{p,q}(M)$

Prop Given hol v.b. E over $\mathbb{C}$ mfd X, 1) $\bar\partial_E^2 = 0$.
2) $\bar\partial_E(fs) = \bar\partial f \wedge s + f\bar\partial_E s$

Pf Directly from the corresponding properties of $\bar\partial$.

Prop If E is any C^∞ complex v.b. over $\mathbb{C}$ mfd X, and we have an operator $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ obeying

- Leibniz: $\bar\partial_E(\alpha \otimes s) = \bar\partial\alpha \otimes s + \alpha \otimes \bar\partial_E s$
- Integrability: $\bar\partial_E^2 = 0$

then $\bar\partial_E$ is induced from a hol. structure on E.

Pf Analog of Newlander-Nirenberg

Rk $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ obeying Leibniz is determined by $\bar\partial_E : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$.

Prop Two $\bar\partial_E, \bar\partial_E'$ both obeying Leibniz have $(\bar\partial_E' - \bar\partial_E)s = \beta(s)$ for $\beta \in \Omega^{0,1}(\text{End } E)$

The action of $\Omega^{0,1}(\text{End } E)$ on $\Omega^{p,q}(E)$ is defined "componentwise" as follows:
If $\beta = \alpha \otimes \varphi$ $\quad \alpha \in \Omega^{0,1}$, $\varphi \in C^\infty(\text{End } E)$ and $s = \gamma \otimes \psi$ $\quad \gamma \in \Omega^{p,q}$, $\psi \in C^\infty(E)$

then $\beta(s) = \underbrace{(\alpha \wedge \gamma)}_{\in \Omega^{p,q+1}} \otimes \underbrace{\varphi(\psi)}_{\in C^\infty(E)} \in \Omega^{p,q+1}(\text{End } E)$

Pf First look at the action on $\Omega^{0,0}(\text{End } E)$.

Define $\beta = \bar\partial_E' - \bar\partial_E$, then it obeys $\beta(fs) = f\beta(s)$

so it's a <u>linear</u> map $E \longrightarrow \Lambda^{0,1} \otimes E$

ie a section of $\Lambda^{0,1} \otimes \operatorname{End} E$ — this is the desired struct for the action on $\Omega^{0,0}(\operatorname{End} E)$.

Then extend to $\Omega^{p,q}(\operatorname{End} E)$ by Leibniz rule $\blacksquare$