Ex Consider the trivial C^∞ line bundle $L \to \Sigma_\tau$.

Sections are just functions on Σ_τ, or doubly periodic functions on $\mathbb{C}$.

Define $\bar\partial_{\mathcal{L}_\alpha} = \bar\partial + \alpha\, d\bar z$, for some $\alpha \in \mathbb{C}$.

This gives a bunch of holomorphic line bundles $\mathcal{L}_\alpha$, with $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta = \mathcal{L}_{\alpha+\beta}$.

Prop $\mathcal{L}_\alpha$ is trivial $\Longleftrightarrow$ $\alpha = \dfrac{-2\pi i(m\tau+n)}{\tau - \bar\tau}$ for some $m,n \in \mathbb{Z}$

Pf Global sections? $(\bar\partial + \alpha\, d\bar z) f = 0$

$$\Longrightarrow \quad f = k(z)\cdot e^{-\alpha \bar z} \qquad k(z) \text{ hol.}$$

or better, $f = h(z)\cdot e^{\alpha(z-\bar z)}$

this requires
$$h(z+1) = h(z)$$
$$h(z+\tau) = h(z)\, e^{-\alpha(\tau-\bar\tau)}$$

One solution to these constraints is $h(z) = \lambda e^{2\pi i m z}$ for some $m \in \mathbb{Z}$.

then $\dfrac{h(z+\tau)}{h(z)} = e^{2\pi i m\tau} = e^{-\alpha(\tau-\bar\tau)}$. So, need $2\pi i(m\tau+n) = -\alpha(\tau-\bar\tau)$.

Moreover, these are the only solutions, (but this needs some analytic tools! Later in the course we'll prove it using Hodge theory.) ∎

Rk 1) Thus, the line bundles $\begin{array}{c}\mathcal{L}_\alpha \\ \downarrow \\ \Sigma_\tau\end{array}$ up to equiv are parameterized by $\alpha \in \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}_\tau) \simeq \Sigma_\tau$

2) They form an honest group! (Unlike Σ_τ itself, which when considered as an abstract $\mathbb{C}$ manifold up to $\simeq$ is not a group.)

3) Are these all the top. trivial line bundles over Σ_τ? Yes. Indeed:

Suppose we had a general
$$\bar\partial_E = \bar\partial + \beta \qquad \beta \in C^\infty(\Omega^{0,1})$$
$$\bar\partial_E' = \bar\partial + \beta'$$

These 2 give equivalent hol structures if $\beta' - \beta = \dfrac{\bar\partial g}{g}$ for some $g : \Sigma_\tau \to \mathbb{C}^\times$. So the Q is: can we reduce β to a constant by such shifts? An analytic question. (Answer: yes.)

4) We'll see a similar story for line bundles on a general compact Kähler manifold.