Hermitian Structures

We discussed f-d vector spaces V, almost $\mathbb{C}$ structures $I : V \to V$
Now, introduce compatible metrics.

Def A positive definite symm. bilinear form g on V is called <u>compatible with I</u> if $g(Iv, Iw) = g(v,w)$ $\forall v,w \in V$
[i.e. $I \in O(V)$]

Def Given (V, g, I) compatible, define $\omega : V \otimes V \to \mathbb{R}$ by $\omega(v,w) = g(Iv, w)$
("fundamental form"), and $h = g - i\omega$.

Prop $\omega \in \Lambda^{1,1} V^*$.

Pf $\omega(v,w) = g(Iv,w) = g(IIv, Iw) = -g(v, Iw) = -g(Iw, v) = -\omega(w,v)$.
$\omega(Iv, Iw) = g(IIv, Iw) = g(Iv, w) = \omega(v,w) \Rightarrow \omega_{\mathbb{C}}$ annihilates $V^{1,0} \otimes V^{1,0}$ or $V^{0,1} \otimes V^{0,1}$ ∎

Rk Any two of (I, g, ω) determine the third.

Prop h is a Hermitian metric on (V, I) viewed as a <u>complex</u> vector space.

Pf $h(v,w) = \overline{h(w,v)}$ ✓

$$\begin{aligned} h(Iv, w) &= g(Iv, w) - i\,\omega(Iv, w) \\ &= g(Iv, w) + i\, g(v,w) \\ &= i\big(g(v,w) - i\,\omega(v,w)\big) \\ &= i\, h(v,w) \ ✓ \end{aligned}$$ ∎

There's another Hermitian metric around:

Def Extend g from V to a Hermitian metric g on $V_{\mathbb{C}}$.

Prop Under $(V, I) \simeq V^{1,0} \subset V_{\mathbb{C}}$, h corresponds to $2g$.
$[v \mapsto \frac{1}{2}(v - iI(v))]$

Pf $h(v,w) = g(v,w) - i\,g(Iv,w)$
$= g(v - iIv, w)$
$= \frac{1}{2} g(v - iIv, w - iIw)$ ∎

Def Given (V, I, ω):
the <u>Lefschetz operator</u> $L : \Lambda^{p,q} V^* \to \Lambda^{p+1,q+1} V^*$ is $\alpha \mapsto \omega \wedge \alpha$

Def Suppose V <u>oriented</u>, $\dim V = n$. Let $\mathrm{vol} \in \Lambda^n(V)$ be $e_1 \wedge \cdots \wedge e_n$ where $\{e_i\}$ is a positively oriented g-orthonormal basis.

Then define <u>Hodge $\star$-operator</u> by

$$\alpha \wedge \star\beta = g(\alpha, \beta)\, \mathrm{vol}.$$

Ex

$$\star(1) = \mathrm{vol} \qquad \begin{array}{ll} \star(e_1) = e_2 \wedge e_3 & \star(e_1 \wedge e_2) = e_3 \\ \star(e_2) = e_3 \wedge e_1 & \star(e_2 \wedge e_3) = e_1 \\ \star(e_3) = e_1 \wedge e_2 & \star(e_3 \wedge e_1) = e_2 \end{array} \qquad \star(e_1 \wedge e_2 \wedge e_3) = 1$$

Prop
- $g(\alpha, \star\beta) = (-)^{k(n-k)} g(\star\alpha, \beta)$ $\qquad \alpha \in \Lambda^k(V)$
- $\star^2 = (-)^{k(n-k)}$ on $\Lambda^k(V)$
- $\star$ is an <u>isometry</u>

Pf Exercise.

Def <u>Dual Lefshetz operator</u> $\Lambda : \Lambda^\bullet(V^*) \to \Lambda^\bullet(V^*)$
is the <u>adjoint</u> to L wrt g.

Prop $\Lambda = \star^{-1} \circ L \circ \star$.

Pf $g(La, b) \cdot \mathrm{vol} = La \wedge \star b = \omega \wedge a \wedge \star b = a \wedge \omega \wedge \star b = g(a, \star^{-1} \circ L \circ \star b) \cdot \mathrm{vol}$ ∎

Def Extend $\star, L, \Lambda$ $\mathbb{C}$-linearly to $V_{\mathbb{C}}$. (Warning: some people define $\star$ to be <u>conjugate</u>-linear)

Prop $\alpha \wedge \star\overline{\beta} = g(\alpha, \beta) \cdot \mathrm{vol}$.

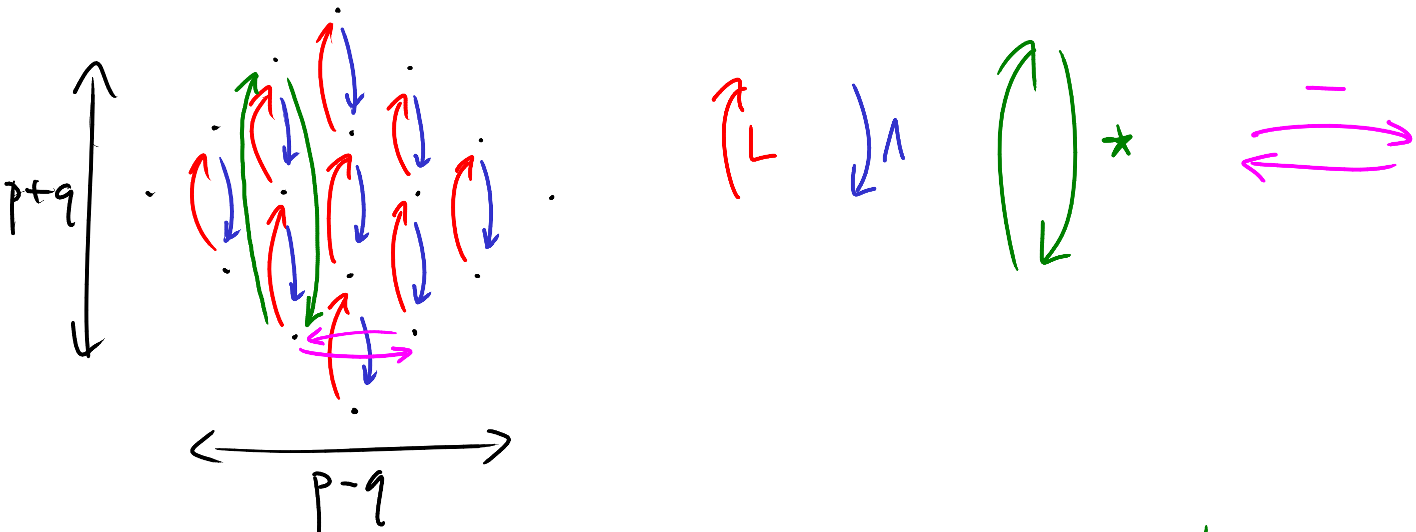
Pf Exercise.

Now say V has almost $\mathbb{C}$ str. I. So $n = 2m$, and there's canonical orientation.
(choose a basis for V as $\mathbb{C}$ vector space and take $e_1 \wedge Ie_1 \wedge \cdots \wedge e_m \wedge Ie_m$)

Prop • $\star : \Lambda^{p,q} V^* \to \Lambda^{m-q,\, m-p} V^*$ • $L : \Lambda^{p,q} V^* \to \Lambda^{p+1,\, q+1} V^*$

• $\Lambda : \Lambda^{p,q} V^* \to \Lambda^{p-1,\, q-1} V^*$

Pf Exercise.



Def $H : \Lambda^\bullet(V_{\mathbb{C}}^*) \to \Lambda^\bullet(V_{\mathbb{C}}^*)$ $\quad H = (k-m)\cdot \mathbb{1}$ on $\Lambda^k(V^*)$

extended $\mathbb{C}$-linearly to $V_{\mathbb{C}}$

Now, $\mathfrak{sl}_2\mathbb{R}$ appears:

Prop a) $[H, L] = 2L$

b) $[H, \Lambda] = -2\Lambda$

c) $[L, \Lambda] = H$

Pf a), b) just say L, Λ raise/lower the degree by 2.

c): induction on dimension. If $V = V_1 \oplus V_2$ (compatible w/ I, g) then

$L = L_1 \oplus 0 + 0 \oplus L_2 : \quad \Lambda^p V_1 \otimes \Lambda^q V_2 \to \Lambda^{p+1} V_1 \otimes \Lambda^q V_2 \oplus \Lambda^p V_1 \otimes \Lambda^{q+1} V_2$

$\Lambda = \Lambda_1 \oplus 0 + 0 \oplus \Lambda_2$.

$H = H_1 \oplus 0 + 0 \oplus H_2$ $\qquad$ Use this to reduce to case $\dim V = 2$.

If $\dim(V) = 2$: take an ON-basis e_1, e_2 with $Ie_1 = e_2$, $Ie_2 = -e_1$.

Then $\omega = e_1 \wedge e_2$ [check!]

$\Lambda^\bullet(V^*) = \mathbb{R} \oplus (e_1\mathbb{R} \oplus e_2\mathbb{R}) \oplus \omega\mathbb{R}$

$L : 1 \mapsto \omega \qquad \Lambda : \omega \mapsto 1 \qquad$ and compute the commutators directly. ∎

Def Let $P^k = \{\alpha \mid \Lambda(\alpha) = 0\} \subset \Lambda^k(V^*)$. (primitive)

$n = 2m$

$(2m = \dim_{\mathbb{R}} V)$

Prop

i) $\Lambda^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i})$, orthogonal for g.

ii) $k > n \Rightarrow P^k = 0$.

iii) $L^{m-k}: P^k \to \Lambda^{2m-k} V^*$ is injective for $k \leq m$

iv) $L^{m-k}: \Lambda^k V^* \to \Lambda^m V^*$ is bijective for $k \leq m$

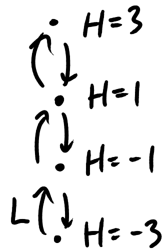
v) If $k \leq m$ then $P^k = \{\alpha \mid L^{m-k+1}\alpha = 0\}$.

Pf Basic tool is $sl_2\mathbb{R}$ rep theory, which we take as a given:

any f-d rep is $\oplus$ of irreps; irreps W_r labeled by $r = 0, 1, 2, \dots$

$$W_r = \bigoplus_{l=0}^{r} V_l, \quad Hw = (2l - r)w \text{ for } w \in V_l, \quad \dim_{\mathbb{R}} V_l = 1$$

e.g. W_3:



So the primitive elements are the ones which are at the bottom: all other elements are obtained by acting with L on these (proves i))

they all have $H \leq 0$ (proves ii))

etc... ∎

This decomp. is compatible w/ bidegree decomp so e.g. $P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}$

Ex $n=2$:

		1			$H=2$
	2		2		$H=1$
1		4		1	$H=0$
	2		2		$H=-1$
		1			$H=-2$
W_0	$W_1^{\oplus 2}$	$W_2 \oplus W_0^{\oplus 3}$	$W_1^{\oplus 2}$	W_0	

Primitive part:

1	3	1
2		2
	1	