Def Call (X,g) <u>Kähler</u> if X is a complex manifold, g a compatible Hermitian metric, and $d\omega = 0$.

Ex $\dim_{\mathbb{C}} X = 1 \Rightarrow$ any Hermitian metric on X is Kähler.

This is a stronger condition than it first appears!

Lemma M almost $\mathbb{C}$, g Hermitian: I integrable $\Leftrightarrow$ $\nabla_{IX} I = I(\nabla_X I)$ $\forall X$

Pf
$$N(X,Y) = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY]$$
(if $\nabla X = \nabla Y = 0$ at p) $= -I(\nabla_Y I)X + I(\nabla_X I)Y - (\nabla_{IX} I)Y + (\nabla_{IY} I)X$ $\quad (*)$

This shows $(\Leftarrow)$. For $(\Rightarrow)$, let $A(X,Y,Z) = g\big[(\nabla_{IX} I - I(\nabla_X I))Y, Z\big]$. $(*) \Rightarrow A(X,Y,Z) = A(Y,X,Z)$.
But $A(X,Y,Z) = -A(X,Z,Y)$ since $I, \nabla_Z I$ commuting skew-adjoint. Together these say $A = 0$. ∎

Prop g Kähler $\Leftrightarrow$ I is <u>parallel</u> wrt the Levi-Civita connection ∇.

Pf $(\Leftarrow)$ $\omega(v,w) = g(Iv,w)$ and $\nabla g = \nabla I = 0 \Rightarrow \nabla \omega = 0$

$(\Rightarrow)$ Set $B(X,Y,Z) = g((\nabla_X I)Y, Z)$. Then
$$\left.\begin{aligned} &B(X,Y,IZ) = B(X,IY,Z) \\ \text{and Lemma says } &B(X,Y,IZ) + B(IX,Y,Z) = 0 \\ \text{thus also } &B(X,IY,Z) + B(IX,Y,Z) = 0. \end{aligned}\right\} (*)$$

Now, $0 = d\omega(X,Y,Z) = (\nabla_X \omega)(Y,Z) + (\nabla_Y \omega)(Z,X) + (\nabla_Z \omega)(X,Y)$ $\quad$ [∇ torsion-free]
$= B(X,Y,Z) + B(Y,Z,X) + B(Z,X,Y)$ $\quad$ [$\nabla g = 0$]
$$\Rightarrow \begin{cases} B(X,Y,IZ) + B(Y,IZ,X) + B(IZ,X,Y) = 0 \\ B(X,IY,Z) + B(IY,Z,X) + B(Z,X,IY) = 0 \end{cases}$$
add these, use $(*) \Rightarrow 2B(X,Y,IZ) = 0$, i.e. $B = 0$, as desired ∎

Rk A Kähler metric involves (I, ω) which each obey an <u>integrability</u> condition, which are <u>compatible</u> $(\omega \in \Omega^{1,1}$, ie $\omega(Iv,Iw) = \omega(v,w))$
and obey a <u>positivity</u> condition:

(equiv, $\omega = \frac{i}{2} h_{ij}\, dz_i \wedge d\bar{z}_j$ with h + def)

Prop X complex $\Rightarrow$
$$\{\text{Kähler metrics on } X\} \xleftrightarrow{\sim} \left\{ \omega \in \Omega^{1,1}(X,\mathbb{R}) \;\middle|\; \begin{aligned} &g(X,Y) = \omega(X,IY) + \omega(IX,Y) \text{ pos def.} \\ &\text{and } d\omega = 0 \end{aligned} \right\}$$

Thus there is a <u>cone</u> of allowed Kähler forms on a given complex manifold X.

One way to produce a closed $(1,1)$–form: write $\omega = i\partial\bar\partial K$ for $K \in \Omega^0$, real.
(In fact every ω is locally of this form — will prove later.)

<u>Ex</u> $\mathbb{C}^n$ is Kähler w/ usual flat metric, Kähler potential $K = \sum_{i=1}^n |z_i|^2$.

But usually, K doesn't exist <u>globally</u>. Rather, have different K in different patches, related by $K' = K + \mathrm{Re}(f(z))$ for f holomorphic.
(then $\partial\bar\partial K' = \partial\bar\partial K$, so ω still globally defined!)

<u>Ex</u> Say $X = \mathbb{CP}^n$. <u>Morally</u>, say $K = \frac{1}{2\pi}\log\|w\|^2 = \frac{1}{2\pi}\log\left(\sum_{j=0}^n |w_j|^2\right)$

This isn't really a well defined function. But, it's OK:
changing $w_i \to f w_i$ shifts $K \to K + \frac{1}{2\pi}\log|f|^2 = K + \mathrm{Re}\left(\frac{1}{\pi}\log f\right)$.

More precisely: take $K_i = \frac{1}{2\pi}\log\left(1 + \sum_{\substack{0\le j\le n \\ i\ne j}} |z_j|^2\right)$ on patch U_i

Then define $\omega = i\partial\bar\partial K$; this is well defined and closed.
Just need to check <u>positivity</u>: in the patch U_0 say, use coords $z_1, \dots, z_n$,

$$\omega = \frac{1}{\left(1+\sum |z_i|^2\right)^2} \sum h_{ij}\, dz_i \wedge d\bar z_j$$

$h_{ij} u$

with $h_{ij} = \left(1 + \sum_i |z_i|^2\right)\delta_{ij} - \bar z_i z_j$

so $h(u,u) = h_{ij} u^i \bar u^j = (1+\|z\|^2)\|u\|^2 - |(z,u)|^2 > \|z\|^2\|u\|^2 - |(z,u)|^2 \geq 0$ [by Cauchy–Schwarz]

Thus we get a Kähler metric on $\mathbb{CP}^n$ [Fubini–Study]

<u>Rk</u> This Kähler structure is not <u>canonical</u> on $\mathbb{P}(V)$ — depends on a choice of Hermitian metric in V!

Rk Our normaliz^n was chosen so that $\int_{\mathbb{CP}^1} \omega = 1$.

$$\left(\int_{\mathbb{C}} \frac{i}{2\pi} \frac{1}{1+|z|^2} dz \wedge d\bar{z} = 2 \int_0^\infty \frac{r\, dr}{(1+r^2)^2} = 1. \right)$$

Ex Torus $\mathbb{C}^n / \Lambda$ is Kähler (Λ any lattice)

Prop If X is Kähler, any cplx submfd $Y \subset X$ is also Kähler.

Pf Just take the restriction of all the structures:

$$I_Y = I_X|_{TY}, \quad g_Y = g_X|_{TY}, \quad \omega_Y = \iota^* \omega_X \qquad (\iota : Y \hookrightarrow X)$$

Compatible on $X \Rightarrow$ compatible on Y ∎

Cor Any smooth projective algebraic variety is Kähler.

"Geometric" interpretation of K:

Def A Hermitian metric h on a C^∞ v.b. $E \to X$ is a Hermitian bilinear form h_x on each fiber E_x, varying smoothly.

Suppose given a hol l.b. $\mathcal{L} \to X$ w/ a Hermitian metric.

Choose local section $s_\alpha \in \Gamma(\mathcal{L}|_{U_\alpha})$ over patch U_α, define K_α by $h(s_\alpha, s_\alpha) = e^{K_\alpha}$. On overlaps, $s_\beta = \psi_{\alpha\beta} s_\alpha$ gives $K_\beta = K_\alpha + \log |\psi_{\alpha\beta}|^2$

Thus, $\omega = i \partial \bar{\partial} K_\alpha$ makes global sense, α-indep. K_α come from the globally defined h on $\mathcal{L}$.

[In pt$^{\text{c}}$, the K we used on $\mathbb{CP}^n$ arose this way, with $\mathcal{L} = \mathcal{O}(-1)$, and Hermitian metric induced from the standard one on $\mathbb{C}^{n+1}$]