Kähler identities

Some preparation:

Def $v \in V$ $\quad \omega \in \Lambda^p V^*$

$v \lrcorner \omega \in \Lambda^{p-1} V^*$ is defined by

$$(v \lrcorner \omega)(v_1, \dots, v_{p-1}) = \omega(v, v_1, \dots, v_{p-1}).$$

$$\left[\text{So } v \lrcorner (\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_{i=1}^{p} (-1)^{i-1} \alpha_i(v)\, \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_p \right]$$

Lemma M Riemannian: Fix an orthonormal basis $\{e_i\}$ for TM and T^*M.

Then $d = \sum e_i \wedge \nabla_{e_i}$, $\quad d^* = -\sum e_i \lrcorner \nabla_{e_i}$

Pf Fix $x \in M$. Then around x we have a Riem. normal coord sys with $e_i = \frac{\partial}{\partial x_i}$ at p.

Then at x, I a multi-index, $\quad d(f\, dx_I) = \sum \frac{\partial f}{\partial x_i} dx_i \wedge dx_I = \sum e_i \wedge \nabla_{e_i}(f\, dx_I)$

$$d^*(f\, dx_I) = (-1)^{n(p+1)+1} \star d \star (f\, dx_I) \qquad p = |I|$$

$$= (-1)^{n(p+1)+1} \sum \frac{\partial f}{\partial x_i} \star [dx_i \wedge \star dx_I] = \underbrace{(-1)^{n(p+1)+1+(p-1)(n-p)}}_{=-1} \sum e_i \lrcorner \nabla_{e_i}(f\, dx_I)$$

Lemma $[v \lrcorner, \alpha \wedge] = (v \lrcorner \alpha) \wedge$ $\quad$ if $|\alpha|$ even.

(if $|\alpha|$ odd, use anticommutator instead)

Pf Exercise.

Lemma If V has compatible (I, g) then $[X \lrcorner, L] = \underset{\substack{\Uparrow \\ V^*}}{IX} \wedge$.

(Using g to identify $V \simeq V^*$.)

Pf The definition $\omega(X, \cdot) = g(\cdot, IX)$

says precisely that $X \lrcorner \omega = IX$ (using g to identify $V \simeq V^*$).

Then use the previous lemma. ∎

Prop X Kähler:

i) $[\bar\partial, L] = [\partial, L] = 0$

$[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$

ii) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$

$[\Lambda, \bar\partial] = -i\partial^*$, $[\Lambda, \partial] = i\bar\partial^*$

iii) $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ and Δ commutes with $\star, \partial, \bar\partial, \partial^*, \bar\partial^*, L, \Lambda$

Pf i) $[\bar\partial, L]\alpha = \bar\partial(\omega\wedge\alpha) - \omega\wedge\bar\partial\alpha = \bar\partial\omega\wedge\alpha = 0 \qquad \alpha\in\Omega^k(X)$

so $[\bar\partial, L] = 0$ and then conjugation gives $[\partial, L] = 0$

$$[\bar\partial^*, \Lambda]\alpha = (-\star\partial\star)(\star^{-1}L\star)\alpha - (\star^{-1}L\underbrace{\star)(-\star}_{=(-1)^k \text{ on } \Omega^{2m-k+1}}\partial\star)\alpha$$

$$= (-\star\partial L\star)\alpha - (-1)^k \underbrace{\star^{-1}}_{= -(-1)^k\star \text{ on } \Omega^{2m-k+3}} L\partial\star\alpha$$

$$= -\star[\partial, L]\star\alpha = 0$$

$\left[\begin{array}{l}\star^2 = (-)^\ell \\ \text{on } \Omega^\ell \\ \text{since } n = 2m\end{array}\right]$

ii) Define $d^c = i(\bar\partial - \partial)$ and $d^{c*} = -\star d^c\star$

then what we need to show is $[L, d^*] = d^c$

But this is (use ON basis)

$$[L, d^*] = -\sum_{i=1}^{2m} [L, e_i \lrcorner \nabla_{e_i}]$$

$$= -\sum_{i=1}^{2m} [L, e_i\lrcorner]\nabla_{e_i} \qquad [\text{because } \nabla_{e_i}\omega = 0]$$

$$= \sum_{i=1}^{2m} Ie_i \wedge \nabla_{e_i} \qquad (\text{using Lemma above})$$

Now $\partial = \sum_{i=1}^{2m} \frac{1}{2}(e_i + i I e_i) \wedge \nabla_{e_i}$

$\bar\partial = \sum_{i=1}^{2m} \frac{1}{2}(e_i - i I e_i) \wedge \nabla_{e_i}$

(projections onto $\Omega^{p+1,q}$ and $\Omega^{p,q+1}$ resp: use the fact ∇_{e_i} preserves $\Omega^{p,q}$ since $\nabla I = 0$)

(NB: tricky sign here — using the metric to identify $T_{\mathbb{C}} \simeq T^*_{\mathbb{C}}$ gives $T^{1,0} \xrightarrow{\sim} \overline{T}^{0,1}$ not $T^{1,0} \xrightarrow{\sim} \overline{T}^{1,0}$!)

So $$\sum_{i=1}^{2m} I e_i \wedge \nabla_{e_i} = i(\bar\partial - \partial) = d^c.$$

iii) $$-i(\bar\partial\partial^* + \partial^*\bar\partial) = \bar\partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\bar\partial$$
$$= \bar\partial\Lambda\bar\partial - \bar\partial\Lambda\bar\partial = 0$$

$[\bar\partial^2 = 0]$

so

$$\Delta = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial)$$
$$= \underbrace{(\partial\partial^* + \partial^*\partial)}_{} + \underbrace{(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)}_{} + (\bar\partial\partial^* + \partial^*\bar\partial) + (\partial\bar\partial^* + \bar\partial^*\partial)$$
$$= \Delta^\partial + \Delta^{\bar\partial} + 0 + 0$$
$$= \Delta^\partial + \Delta^{\bar\partial}$$

and $$-i\Delta^\partial = -i(\partial\partial^* + \partial^*\partial) = \partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\partial$$
$$= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial$$
$$= \partial\Lambda\bar\partial + \bar\partial\partial\Lambda - \Lambda\partial\bar\partial - \bar\partial\Lambda\partial$$
$$= [\partial, \Lambda]\bar\partial + \bar\partial[\partial, \Lambda]$$
$$= -i\Delta^{\bar\partial}$$

so $\Delta^\partial = \Delta^{\bar\partial} = \frac{1}{2}\Delta$

To prove commutation, e.g.

use $[\Lambda, d] = -d^{c*}$ and $[\Lambda, d^*] = 0$

use $\{d^*, d^{c*}\} = 0$

$$\Lambda\Delta = \Lambda d d^* + \Lambda d^* d = d\Lambda d^* - d^{c*}d^* + d^*\Lambda d$$
$$= d d^*\Lambda - d^{c*}d^* - d^* d^{c*} + d^* d\Lambda = \Delta\Lambda$$

Another point of view on ii): It's an identity involving only <u>first derivatives</u>, and Kähler manifolds admit "normal coords" around any point:

<u>Prop</u> If X is Kähler and $x_0 \in X$ then there are coordinates around x_0 for which

$$\omega = i \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + O\left(\sum |z_i|^2\right)$$

i.e.

$$\omega = i \sum h_{ij} \, dz_i \wedge d\bar{z}_j \quad \text{with} \quad h_{ij}(0) = \delta_{ij}$$

$$dh_{ij}(0) = 0$$

This means we could prove ii) just by computations in $\mathbb{C}^n$.