Now revisit the study of line bundles over $\Sigma_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$ which are topologically trivial.

We described them via $\bar\partial_{\mathcal{L}} = \bar\partial + \alpha \qquad \alpha \in \Omega^{0,1}(\Sigma_\tau)$.

Recall we claimed that for any $\alpha \in \Omega^{0,1}(\Sigma_\tau)$
there exists $\beta \in \Omega^{0,0}(\Sigma_\tau)$ such that $\alpha + \bar\partial\beta$ is a constant multiple of $d\bar z$.

Equivalently, we claimed that $\dim H^{0,1}(\Sigma_\tau) = 1$.

Now we see this as a consequence of, e.g., $\dim \mathcal{H}^{0,1}(\Sigma_\tau) = 1$

(since $\Delta(f\, d\bar z) = \Delta f\, d\bar z$, and on compact manifolds $\Delta f = 0 \Leftrightarrow f$ constant)

We also claimed that $\exists$ a hol function f with $f(z+1) = f(z)$, $f(z+\tau) = c \cdot f(z)$, no zeroes,
only if $c = e^{2\pi i k\tau}$ for some $k \in \mathbb{Z}$. This we can prove just by taking $\ell(z) = \log f(z)$ $(\exists!)$

$\ell(z+1) = \ell(z) + 2\pi i k, \quad \ell(z+\tau) = \ell(z) + \log c + 2\pi i n$

then consider $g(z) = \ell(z) - 2\pi i k z$

$g(z+1) = g(z), \quad g(z+\tau) = g(z) + \log c + 2\pi i(n + k\tau)$

But then $g'(z)$ is doubly periodic, so $g'(z)$ is constant, and $g(z+1) = g(z)$, so $g(z)$ is constant.

So using
1) Hodge theory
2) local surjectivity of exp

we classified top. triv. line bundles over torus.

Let's see how this works in much more generality..