Holomorphic line bundles, redux

Let $\mathrm{Pic}(X)$ be the gp of hol. line bundles over X, with mult. given by $\otimes$.

Prop $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^\times)$.

Pf Sketch Fix a covering $\mathcal{U}$ of X by polydiscs. ($\exists$?)

Use description via transition functions to get a map

$H^1(X, \mathcal{O}^\times) \to \mathrm{Pic}(X)$. Well defined b/c changing by Čech

coboundary doesn't change the line bundle. Injective b/c if

the line bundle is trivial it gives a triv^n of the transition func.

Surjective b/c every line bundle over polydisc is trivial (prove using characteriz^n via $\bar\partial$ ops) ∎

Use $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$

to get $\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots$

Thus loosely speaking, $\mathrm{Pic}(X)$ has a "cts part" in $H^1(X, \mathcal{O}) \simeq H^{0,1}(X)$

plus a "discrete part" in $H^2(X, \mathbb{Z})$. In ptc, $H^{0,1}(X)$ surjects onto subgp

$\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$ with $c_1(\mathcal{L}) = 0$. The kernel of $H^{0,1}(X) \to \mathrm{Jac}(X)$ is

$\mathrm{Im}(H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}))$. That's injective when X compact.

[To see this, go back a few steps, to $H^0(X, \mathcal{O}) \xrightarrow{\exp} H^0(X, \mathcal{O}^\times)$ which is surjective since we can always take the log of a constant function]

Thus we proved

Prop X compact $\Rightarrow$ $\mathrm{Jac}(X) \simeq \dfrac{H^{0,1}(X)}{H^1(X, \mathbb{Z})}$.