Characteristic Classes

Want to extract topological information from a complex vector bundle.

Short-term reasons: this topological info. will occur naturally in

1) HRR formula,
2) Yau's Thm.

So: let E be C^∞ v.b. w/ connection ∇.

Have $F_\nabla \in \Omega^2(\operatorname{End} E)$ with $\nabla F_\nabla = 0$.

How to make s.t. digestible from it?

First, linear algebra:

Def Say $V = gl(r, \mathbb{C}) = \{r \times r \text{ matrices}\}$

$$P : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{C} \quad \text{multilinear symmetric}$$

P is <u>invariant</u> if

$$(*) \quad P(CB_1C^{-1}, \ldots, CB_kC^{-1}) = P(B_1, \ldots, B_k) \quad \forall B_1, \ldots, B_k \in V, \; C \in GL(r, \mathbb{C}).$$

Rk • Could similarly replace $GL(r,\mathbb{C}) \rightsquigarrow$ any complex Lie gp G
$V \rightsquigarrow$ any G-rep
and say $P \in S^k(V)^*$ <u>invariant</u> if it's fixed by G action.

• In present case such P are all <u>generated by</u> $P(B_1, \ldots, B_k) = \operatorname{Tr}(B_1 \cdots B_k)$

Lemma P is invariant $\iff \displaystyle\sum_{j=1}^{k} P(B_1, \ldots, B_{j-1}, [B, B_j], B_{j+1}, \ldots, B_k) = 0$.

Pf In Lie theory this is the stmt that a rep of G is trivial iff the corresponding rep of $\mathfrak{g} = \operatorname{Lie}(G)$ is trivial. To prove it "by hand,"

($\Rightarrow$) take $C = e^{tB}$ and differentiate $(*)$ at $t=0$.

($\Leftarrow$) take $C = e^{tB}$ and look at a general G-rep in which $\mathfrak{g}$ acts trivially (just to simplify <u>notation</u>):

$$Cv - v = e^{tB}(v) - v = \int_0^t dt' \; B\left[e^{t'B}(v)\right] = 0$$

But such C generate the whole group G. ∎

Now, our strategy: apply such P to $(F_\nabla, \dots, F_\nabla)$ to get honest forms on M.

Def/Prop If P invariant k-multilinear symm form on $V = \mathfrak{gl}(r, \mathbb{C})$,
E C^∞ rank-r v.b.,
$m = i_1 + \cdots + i_n$,

$\exists$ natural

$$P : \left(\Lambda^{i_1} M \otimes \operatorname{End}(E)\right) \times \cdots \times \left(\Lambda^{i_n} M \otimes \operatorname{End} E\right) \longrightarrow \Lambda^m_{\mathbb{C}} M$$

given in any local triv. of E by

$$P(\alpha_1 \otimes t_1, \dots, \alpha_n \otimes t_n) = (\alpha_1 \wedge \cdots \wedge \alpha_n)\, P(t_1, \dots, t_n).$$

Pf Just have to check it's well defined, which follows from invariance of P. ∎

Lemma $$dP(\gamma_1, \dots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} |\gamma_\ell|} P(\gamma_1, \dots, \nabla(\gamma_j), \dots, \gamma_k).$$

(∇ means the induced conn. in $\operatorname{End}(E)$)

Pf Locally, $\nabla = d + A$.
Acting on $\operatorname{End}(E)$, this becomes $\nabla = d + [A, \cdot]$. (Exercise)

$$dP(\gamma_1, \dots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} |\gamma_\ell|} P(\gamma_1, \dots, d\gamma_j, \dots, \gamma_k)$$

this is just the usual formula for d acting on a wedge product: $\operatorname{End}(E)$ plays no role

$$= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} |\gamma_\ell|} P(\gamma_1, \dots, \nabla\gamma_j - [A, \gamma_j], \dots, \gamma_k)$$

and the term involving $[A, \cdot]$ vanishes by invariance of P. ∎

So in particular,

Cor $\widetilde{P}(F_\nabla) = P(F_\nabla, \ldots, F_\nabla) \in \Omega^{2k}_{\mathbb{C}}(M)$ is closed.

Lemma If ∇, ∇' two different conn. on E,

$$[\widetilde{P}(F_\nabla)] = [\widetilde{P}(F_{\nabla'})]$$

Pf $\nabla' = \nabla + a \qquad a \in \Omega^1(M, \operatorname{End} E)$

$F_{\nabla'} = F_\nabla + \nabla(a) + a \wedge a$

It's enough to show $\left.\frac{d}{dt}\right|_{t=0} [\widetilde{P}(F_{\nabla + ta})] = 0$

Now, $\left.\frac{d}{dt}\right|_{t=0} \widetilde{P}(F_{\nabla+ta}) = k P(F_\nabla, \ldots, F_\nabla, \nabla(a))$

$= k\, dP(F_\nabla, \ldots, F_\nabla, a)$ (using Bianchi and lemma above) ∎

This is how we'll produce characteristic classes.

Def Define $\widetilde{P}_k$ by

$$\det(\mathbb{1} + B) = 1 + \widetilde{P}_1(B) + \cdots + \widetilde{P}_r(B)$$

Homogeneous polynomials.

ex if $r=2$,

$\det(\mathbb{1}+B) = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$

$= 1 + a + d + ad - bc$

so $\widetilde{P}_1(B) = \operatorname{tr} B$,

$\widetilde{P}_2(B) = \det B = \frac{1}{2}\left(\operatorname{tr}(B)^2 - \operatorname{tr}(B^2)\right)$

Chern forms: $c_k(E, \nabla) = \widetilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right)$

Chern classes: $c_k(E) = [c_k(E,\nabla)] \in H^{2k}(X, \mathbb{C})$ for any ∇ on E.

Total Chern class: $c(E) = \sum_{k=0}^{r} c_k(E) = [\det(\mathbb{1} + F_\nabla)]$

Rk If E admits a holomorphic structure, then using Chern conn, $F_\nabla \in \Omega^{1,1}(\operatorname{End} E)$

so $c_k(E,\nabla) \in \Omega^{k,k}(\operatorname{End} E)$. If X compact Kähler, $\Rightarrow c_k(E) \in H^{k,k}(E) \subset H^{2k}(X, \mathbb{C})$.

Obstruction to existence of holomorphic structures on arbitrary C^∞ bundles!

Prop (Whitney product)

If $E = E_1 \oplus E_2$ with $\nabla = \nabla_1 \oplus \nabla_2$, then $c(E,\nabla) = c(E_1,\nabla_1)\cdot c(E_2,\nabla_2)$

Pf Use $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$ and

$$\det\left[\left(\mathbb{1} + \frac{i}{2\pi}F_{\nabla_1}\right) \oplus \left(\mathbb{1} + \frac{i}{2\pi}F_{\nabla_2}\right)\right] = \det\left(\mathbb{1} + \frac{i}{2\pi}F_{\nabla_1}\right)\det\left(\mathbb{1} + \frac{i}{2\pi}F_{\nabla_2}\right)$$

Cor $c(E_1 \oplus E_2) = c(E_1)\cdot c(E_2)$

$$\left[\begin{array}{ll} \text{e.g.} & c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2) \\ & c_2(E_1 \oplus E_2) = c_2(E_1) + c_2(E_2) + c_1(E_1)c_1(E_2) \\ & \quad \vdots \end{array}\right]$$

Can also organize the information in different ways, e.g.:

Def Define $\tilde{P}_k$ by

$$\mathrm{tr}(e^B) = \tilde{P}_0(B) + \tilde{P}_1(B) + \cdots \qquad \left(\text{i.e. } \tilde{P}_k(B) = \frac{1}{k!}\mathrm{tr}(B^k)\right)$$

Then define $ch_k(E,\nabla) = \tilde{P}_k\left(\frac{i}{2\pi}F_\nabla\right) \in \Omega^{2k}_{\mathbb{C}}(M)$.

Chern character: $ch_k(E) = [ch_k(E,\nabla)] \in H^{2k}(M,\mathbb{C})$.

Total Chern char: $ch(E) = \sum_i ch_k(E) = \left[\mathrm{tr}\, e^{\frac{i}{2\pi}F_\nabla}\right]$

Prop If $E = E_1 \otimes E_2$ w/ $\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2$,

$$ch(E,\nabla) = ch(E_1,\nabla_1)\cdot ch(E_2,\nabla_2).$$

Pf Use $F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$

$$\mathrm{tr}\left(e^{\frac{i}{2\pi}F_\nabla}\right) = \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}} \otimes e^{\frac{i}{2\pi}F_{\nabla_2}}\right) = \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}}\right)\mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_2}}\right) \quad \blacksquare$$

$$\left[\frac{d}{dt}e^{tA}\otimes e^{tB} = (A\otimes 1 + 1\otimes B)(e^{tA}\otimes e^{tB})\right]$$

Cor $ch(E_1 \otimes E_2) = ch(E_1)\cdot ch(E_2)$

Prop $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$.

Pf Exercise.

Prop X compact Kähler $\Rightarrow$
using the inclusion $H^k(X, \Omega^k) \simeq H^{k,k}(X) \subset H^{2k}(X, \mathbb{C})$
we have
$$ch_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k tr\left[A(E)^k\right]$$
[where $A(E) \in H^1(X, \mathrm{End}\, E \otimes \Omega^1)$
induces $A(E)^k \in H^k(X, \mathrm{End}\, E \otimes \Omega^k)$ by composition in $\mathrm{End}(E)$
and wedge product in $\Omega^\cdot$]

Pf Use $[F_\nabla] = A(E)$. ∎

If $M = X$ (almost) cplx, we can apply this stuff to its tangent bundle:

Def $c(X) = c(T^{1,0}X)$.

(Not $T_{\mathbb{C}}X$ — that would give instead "Pontryagin classes")

This has interesting information about X. [e.g. $\int_X c_m(X) = \chi(X)$ (Euler characteristic)]

Def E hol v.b. of rank r: $\det E = \Lambda^r(E)$. (a line bundle.)

Prop If E has rank r, then $c_1(E) = c_1(\det E)$

Pf $c_1(E) = \frac{i}{2\pi}[\operatorname{Tr} F_\nabla]$

The endomorphism $F_\nabla \in \Omega^2(\operatorname{End} E)$

induces $\Lambda^r(F_\nabla) \in \Omega^2(\operatorname{End} \Lambda^r(E))$

and $\operatorname{Tr} \Lambda^r(F_\nabla) = \operatorname{Tr} F_\nabla$

[Because the same identity holds for *matrices*: $A \in \operatorname{End} V$ induces $\Lambda^r(A) \in \operatorname{End} \Lambda^r(V)$, and if A is *diagonalizable* $Av_i = \lambda_i v_i$ then $\Lambda^r(A)(v_1 \wedge \cdots \wedge v_r) = (\lambda_1 + \cdots + \lambda_r)(v_1 \wedge \cdots \wedge v_r)$] $\blacksquare$

(You can prove lots of identities this way. One way of thinking about it is that you're formally allowed to pretend (E, ∇) is $\oplus$ of line bundles (L_i, ∇_i). Indeed, can construct some $N \xrightarrow{\pi} M$ with $H^*(M) \hookrightarrow H^*(N)$, and s.t. $\pi^*(E)$ really does split. In this way one can prove stronger statements as well, for *integral* characteristic classes)