Char. classes for K3

Given a smooth hypersurface $Y \subset X$, have a seq. of hol v.b. on Y:

$$0 \to T_{hol}Y \to T_{hol}X|_Y \to \underset{\mathcal{O}(Y)|_Y}{\overset{\|}{NY}} \to 0$$

It's split in C^∞ category $\Rightarrow$ $c(NY)\cdot c(TY) = c(TX|_Y)$ $\quad i: Y \hookrightarrow X$

i.e. $i^* c(\mathcal{O}(Y))\cdot c(TY) = i^* c(TX)$

and $c(E) = 1 + c_1 + c_2 + c_3 + \cdots$

$c(E)^{-1} = 1 - c_1 + (c_1^2 - c_2) + \cdots$ $\qquad \left(\frac{1}{1+x} = 1 - x + x^2 + \cdots\right)$

$$\Rightarrow c(TY) = i^*\left[(1 + c_1(TX) + c_2(TX) + \cdots)(1 - c_1(\mathcal{O}(Y)) + c_1(\mathcal{O}(Y))^2 + \cdots)\right]$$

So $c_1(TY) = i^*(c_1(TX) - c_1(\mathcal{O}(Y)))$

$c_2(TY) = i^*(c_2(TX) - c_1(TX)\cdot c_1(\mathcal{O}(Y)) + c_1(\mathcal{O}(Y))^2)$

Now suppose Y is a degree-d hypersurface in $X = \mathbb{CP}^n$

Defined by the vanishing of a deg-d homogeneous poly. $P_d(z_0, \ldots, z_n) = 0$.

P_d is a section of $\mathcal{O}(d) \to \mathbb{CP}^n$. So, $\mathcal{O}(Y) = \mathcal{O}(d)$.

Thus

$$\begin{aligned} c_1(TY) &= i^*(c_1(TX) - c_1(\mathcal{O}(Y))) \\ &= (n+1-d)\, i^*\mathcal{O}(1). \end{aligned}$$

Important special case: take $\underline{d = n+1}$. Then $c_1(T_{hol}Y) = 0$.
(Calabi-Yau condition, as we'll see.)

e.g. $n=2,\ d=3$: cubic curve in $\mathbb{CP}^2$ (genus 1 curve — flat)
$n=3,\ d=4$: quartic surface in $\mathbb{CP}^3$ ("K3 surface")
$n=4,\ d=5$: quintic — — $\mathbb{CP}^4$ ("quintic CY 3-fold" — first example of mirror symmetry)

For K3 surface:

$$c_2(T_{hol}Y) = i^*\left(c_2(T_{hol}\mathbb{CP}^3)\right)$$

And $c_2(T_{hol}\mathbb{CP}^3) = 6\, c_1(\mathcal{O}(1))^2$

[using $0 \to \mathcal{O} \to \mathcal{O}(1)^4 \to T_{hol}\mathbb{CP}^3 \to 0$
$c(T_{hol}\mathbb{CP}^3) = c(\mathcal{O}(1))^4$
$= 1 + 4c_1(\mathcal{O}(1)) + 6c_1(\mathcal{O}(1))^2 + \cdots$]

So: $$\int_Y c_2(T_{hol}Y) = 6\int_Y c_1(\mathcal{O}(1))^2 = 6\int_{\mathbb{CP}^3} c_1(\mathcal{O}(1))^2 \wedge c_1(\mathcal{O}(4))$$

[Since $\mathcal{O}(Y) = \mathcal{O}(4)$ and $c_1(\mathcal{O}(Y))$ is Poincaré dual to Y]

$$= 24\int_{\mathbb{CP}^3} c_1(\mathcal{O}(1))^3$$

$$= 24$$