Serre duality

X compact Kähler, $\dim_{\mathbb{C}} X = n$.

We've noted before that the pairing
$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}$$
$$(\alpha \;,\; \beta) \longmapsto \int_X \alpha \wedge \beta$$

is <u>nondegenerate</u>: $H^{p,q}(X) \simeq H^{n-p,n-q}(X)^*$.

Or, said otherwise, $H^q(X, \Omega^p) \simeq H^{n-q}(X, \Omega^{n-p})^*$

In particular, $H^q(X, \mathcal{O}) \simeq H^{n-q}(X, K)^*$

To prove it, we used the map
$$\bar{\star} : \Omega^{p,q}(X) \longrightarrow \overline{\Omega^{n-p,n-q}(X)} \quad \text{and} \quad \int \alpha \wedge \bar{\star}\alpha \neq 0.$$
$$\alpha \longmapsto \overline{\star\alpha}$$

This has a useful <u>extension</u>: given a Hermitian vector bundle (E, h) we have
$$\bar{\star}_E : \Omega^{p,q}(X, E) \longrightarrow \overline{\Omega^{n-p,n-q}(X, E^*)}$$
$$\alpha \otimes s \longmapsto \overline{\star\alpha} \otimes h(s)$$

where we view h as an iso $E \simeq \overline{E^*}$

Then we define e.g. $\bar{\partial}_E^* = -\bar{\star}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{\star}_E$

(NB, when E is trivial this does agree with our previous $\bar{\partial}^* = -\star \circ \partial \circ \star$)
$$\Delta_{\bar{\partial},E} = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$$

and develop the whole $\bar{\partial}$-part of Hodge theory as before: in particular,
$$H^{p,q}(X, E) = \mathcal{H}^{p,q}_{\bar{\partial}}(E)$$

Then get a similar kind of duality:
$$H^{p,q}(X, E) \simeq H^{n-p,n-q}(X, E^*)^*$$
i.e. $\quad H^q(X, \Omega^p \otimes E) \simeq H^{n-q}(X, \Omega^{n-p} \otimes E^*)^*$ and thus,

<u>Prop</u> X compact Kähler: $H^q(X, E) \simeq H^{n-q}(X, K \otimes E^*)^*$ (<u>Serre duality</u>)

Ex $X = \text{torus}$: then $K = \text{trivial}$

so get $$H^0(X,E) \simeq H^1(X,E^*)^*$$

$$h^0(E) = h^1(E^*)$$

(This is another explanation of our observation that $h^1(\mathcal{O}) = 1$.)

In general Serre duality means we "only have to understand half of the cohomology to understand all of it."