Vector bundles over $\mathbb{CP}^1$

Thm Any hol v.b. over $\mathbb{CP}^1$ is $\oplus$ of $\mathcal{O}(a_i)$, w/ a_i uniquely determined (up to perm.)

Pf Induction. Idea: try to pick off the largest a_i.

Let $a_1 = \max\left(a : h^0(E \otimes \mathcal{O}(-a)) > 0\right)$

[this $\exists$: for $a \ll 0$, use Serre vanishing to see $h^1 = 0$, HRR to see $\chi \neq 0$ so $h^0 \neq 0$;
for $a \gg 0$, use Serre vanishing to see $h^0 = 0$]

A section of $E \otimes \mathcal{O}(-a_1)$ means a map $\rho : \mathcal{O}(a_1) \to E$.

If it vanished somewhere then we could build a map $\mathcal{O}(a_1 + 1) \to E$, contradicting maximality. So, ρ is injective.

Define $E_1 = E / \mathrm{Im}(\rho)$. E_1 is a vector bundle.

$$0 \to \mathcal{O}(a_1) \to E \to E_1 \to 0$$

By induction $E_1 \simeq \bigoplus_{i=2}^{n} \mathcal{O}(a_i)$.

$$0 \to \mathcal{O}(a_1) \to E \to \bigoplus_{i=2}^{n} \mathcal{O}(a_i) \to 0$$

Why is it split?

First, $a_1 \geq a_i$. (Because otherwise $H^0(E \otimes \mathcal{O}(-a_1 - 1)) \neq 0$,

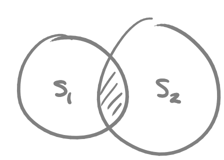
using l.e.s. for $0 \to \mathcal{O}(-1) \to E \otimes \mathcal{O}(-a_1 - 1) \to \bigoplus_{i=2}^{n} \mathcal{O}(a_i - a_1 - 1) \to 0$:

$$\cancel{H^0(\mathcal{O}(-1))}^{\,0} \to H^0(E \otimes \mathcal{O}(-a_1-1)) \to \oplus H^0(E \otimes \mathcal{O}(a_i - a_1 - 1)) \to \cancel{H^1(\mathcal{O}(-1))}^{\,0}$$

But the obstruction to being split lies in $H^1\left(\mathrm{Hom}\left(\bigoplus_{i=2}^{n} \mathcal{O}(a_i), \mathcal{O}(a_1)\right)\right)$

$$= H^1\left(\bigoplus_{i=2}^{n} \mathcal{O}(\underbrace{a_1 - a_i}_{\geq 0})\right)$$

$$= 0.$$



$S_1 - S_2 \in \mathrm{Hom}(-,-)$

So $E \simeq \left[\bigoplus_{i=2}^{n} \mathcal{O}(a_i)\right] \oplus \mathcal{O}(a_1)$ as desired. ∎