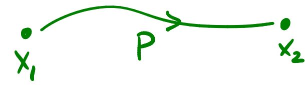
Special Holonomy

Def ∇ any connection in a bundle $V \to M$, $x \in M$:

Parallel transports $\mathbb{P}_{\nabla,p} \in \mathrm{Hom}(V_{x_1}, V_{x_2})$



$$\mathrm{Hol}_\nabla(x) = \{ \mathbb{P}_{\nabla,p} : p \text{ piecewise smooth path in } M \text{ from } x \text{ to } x \} \subset GL(V_x)$$

$$\mathrm{Hol}^\circ_\nabla(x) = \{ \quad " \quad " \quad " \quad \underbrace{"}_{\text{null-homotopic}} \quad " \quad " \quad " \quad " \quad " \quad " \}$$

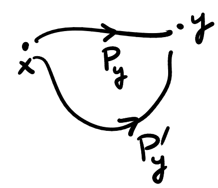
$\mathrm{Hol}_\nabla(x)$ is a <u>Lie subgroup</u> of $GL(V_x)$. $\mathrm{Hol}_\nabla(x') \simeq \mathrm{Hol}_\nabla(x)$.

∇ flat $\iff$ $\mathrm{Hol}^\circ_\nabla(x)$ trivial.

Prop (Holonomy principle) M connected, ∇ connection in $V \to M$, $x \in M$, $v \in V_x$:

$\forall g \in \mathrm{Hol}_\nabla(x)\ gv = v \iff v$ can be extended to ∇-flat section of V over M.

<u>Pf</u> ($\Rightarrow$) use $\mathbb{P}$ to build the extension: $s(y) = \mathbb{P}_{\nabla,p_y}(v)$



$$\mathbb{P}_{\nabla,p_y'}(v) = \mathbb{P}_{\nabla,p_y} \circ \underbrace{\mathbb{P}_{\nabla,p_y^{-1}} \cdot \mathbb{P}_{\nabla,p_{y'}}}_{\in \mathrm{Hol}_\nabla(x)}(v) = \mathbb{P}_{\nabla,p_y}(v)$$

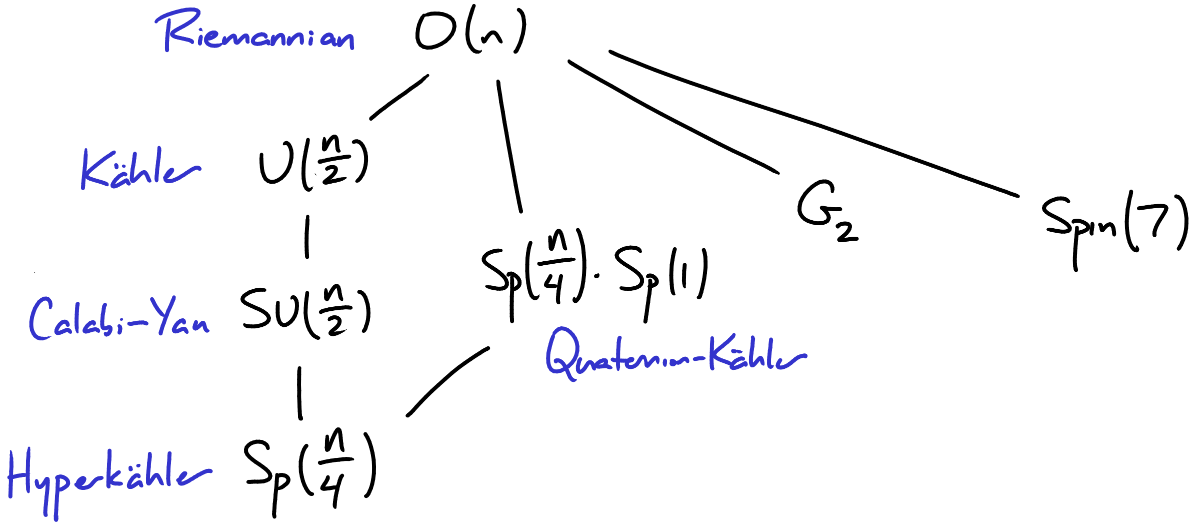
($\Leftarrow$) easy. ∎

Ex

- If ∇ is the Levi-Civita connection for some Riem. metric, $\nabla g = 0 \Rightarrow$ $\mathrm{Hol}_\nabla(x)$ acting on $T^* \otimes T^*$ preserves g. Equivalently, $\mathrm{Hol}_\nabla(x) \subset O(T_x) \subset GL(T_x)$.
- If X is Kähler and ∇ is Levi-Civita, $\nabla g = \nabla \omega = 0 \Rightarrow \nabla h = 0 \Rightarrow$ then $\mathrm{Hol}_\nabla(x) \subset U(T^{1,0}_x X) \subset O(T_x X) \subset GL(T_x)$

This is an instance of "special holonomy".

The possible $\simeq$ types of $\mathrm{Hol}_\nabla(x)$ for ∇ Levi-Civita have been <u>classified</u>: as long as M is <u>irreducible</u> (not locally a direct Π) and also not locally a symmetric space the only possibilities are [Berger, Simons]:


Riemannian $O(n)$
Kähler $U(\frac{n}{2})$
Calabi-Yan $SU(\frac{n}{2})$
Hyperkähler $Sp(\frac{n}{4})$
$Sp(\frac{n}{4}) \cdot Sp(1)$
Quaternion-Kähler
G_2
$Spin(7)$