Strominger-Yau-Zaslow picture of CY manifolds

Recall:

Def X symplectic, $L \subset X$ submfd:
L is Lagrangian if $i^*\omega = 0$ and $\dim L = \frac{1}{2}n$.

If X is CY then X has $\Omega \in H^0(K)$ nowhere vanishing.

Def X CY, $L \subset X$ submanifold:
L is special Lagrangian with phase ϑ if L is Lagrangian and
$i^*\left[\mathrm{Im}(e^{i\vartheta}\Omega)\right] = 0$. $\quad (i : L \hookrightarrow X)$

Rk Prototype: $X = \mathbb{C}^n$, $\omega = i\sum dz_i \wedge d\bar{z}_i$, $\vartheta = 0$, $\Omega = dz_1 \wedge \cdots \wedge dz_n$, $L = \mathbb{R}^n \subset \mathbb{C}^n$.

NB, in this case $i^*\left[\mathrm{Re}(e^{i\vartheta}\Omega)\right] = \mathrm{vol}_L$.

More generally: suppose we normalize $|\Omega \wedge \overline{\Omega}| = \mathrm{vol}$.

Then along any submanif. L of dimension $\frac{1}{2}\dim X$ we have
$|\mathrm{Re}(e^{i\vartheta}\Omega)| \leq \mathrm{vol}_L$. [Harvey-Lawson]

So $\mathrm{vol}(L) \geq \int_L |\mathrm{Re}(e^{i\vartheta}\Omega)| \qquad \left(\text{Or, optimizing over } \vartheta,\ \mathrm{vol}(L) \geq \left|\int_L \Omega\right|.\right)$

Special Lagrangian L saturate this inequality (so in particular they are volume-minimizing: any L' with $[L] = [L']$ has $\mathrm{vol}(L') \geq \mathrm{vol}(L)$).

cf. the Kähler form on any Kähler X: $d\omega = 0$, $\frac{\omega^n}{n!} \leq \mathrm{vol}$.
So every Y has $\mathrm{vol}(Y) \geq \int_Y \frac{\omega^n}{n!}$

Holomorphic Y saturate this inequality.

Strominger-Yau-Zaslow's proposal: every Calabi-Yau manifold X has

$$\pi : X \longrightarrow S^n$$

where a generic fiber $\pi^{-1}(u)$ is a special Lagrangian n-torus.

Gross-Wilson / Kontsevich–Soibelman–Todorov:

this structure gives a useful picture of what Ricci-flat Kähler metrics on X actually "look like": as one deforms the complex structure of X to a "large complex structure point," while holding the diameter of X fixed, the metric on X collapses to a metric on S^n.

What do we mean by "collapsing"? Gromov-Hausdorff sense:

Def If X, Y are metric spaces and $\exists\ f: X \to Y$, $g: Y \to X$ (not nec. cts) with $|d(x_1,x_2) - d(f(x_1), f(x_2))| < \varepsilon \quad \forall x_1, x_2 \in X$

and $|d(x, g\circ f(x))| < \varepsilon \quad \forall x \in X$

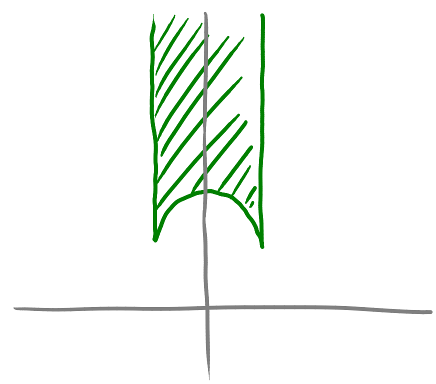
and similarly with X and Y reversed,

then say "$d_{GH}(X,Y) \leq \varepsilon$."

Define $d_{GH}(X,Y)$ to be infimum of all such ε.

In dimension $n=1$, we can really understand what this means:

consider the torus $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$



Recall inequivalent X_τ are labeled by $\tau \in \mathbb{H}/SL(2,\mathbb{Z})$.

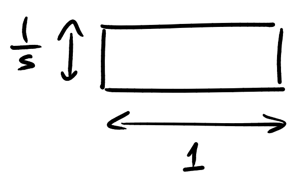
Only one infinite "end": $\tau \to i\infty$.

This will be the "large $\mathbb{C}$ structure" limit.

Say $\tau = is$, $s \in \mathbb{R}_+$, $z = x+iy$. $\quad (x,y) \sim (x+1, y) \sim (x, y+s)$.

Ricci-flat Kähler metric $\quad g = \frac{1}{s^2}(dx^2 + dy^2)$

has diameter 1 (for $s > 1$)



Think of it as a circle fibration over a circle. As $s \to \infty$, fibers collapse, metric approaches a circle in G-H sense.

How about $n=2$? For 4-tori, similar story to $n=1$.

For K3, more interesting.

First, let's construct the desired torus fibrations over S^2.

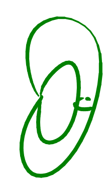
Do it complex-analytically (even though the fibers were supposed to be Lagrangian: we'll later see that they are, but in a different $\mathbb{C}$ str)

So, view S^2 as $\mathbb{CP}^1$. Over each point $u \in \mathbb{CP}^1$, we want to put a torus (genus 1 curve). Try representing that as a cubic curve,

$\Sigma_u = \{y^2 z = x^3 + A(u) x z^2 + B(u) z^3\} \subset \mathbb{CP}^2$. X will be the union of the Σ_u.

What should $A(u)$ and $B(u)$ be? Let's try to get some intuition. We want X to be a K3 surface, so $\chi(X) = 24$.

Generic Σ_u is a smooth genus 1 curve $\Rightarrow$ contributes 0 to $\chi(X)$.



At a simple zero of $\Delta(u) = 27 B(u)^2 - 4 A(u)^3$, Σ_u is singular.
$[y^2 = P_3(x)$ where P has 1 double, 1 simple root$]$ Let $D = \mathrm{div}(\Delta)$.
Each such singular fiber has $\chi = 1$ $\Rightarrow$ contributes 1 to $\chi(X)$.

So, we want $|D| = 24$, i.e. globally $\Delta(u)$ a sec. of $\mathcal{O}(24)$. Easiest way to get that:

$B(u)$ a section of $\mathcal{O}(8)$, $A(u)$ a section of $\mathcal{O}(12)$.

So:

Write $X = \{y^2 z = x^3 + A(u) x z^2 + B(u) z^3\} \subset \left[\begin{matrix}\mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O} \\ \downarrow \\ \mathbb{CP}^1\end{matrix}\right] / \mathbb{C}^\times$

where $A(u)$ is a section of $\mathcal{O}(8)$ x is valued in $\mathcal{O}(4)$
$B(u)$ " " " " $\mathcal{O}(12)$ y " " " $\mathcal{O}(6)$
z " " " $\mathcal{O}(0)$

and $\mathbb{C}^\times$ acts by $(x,y,z) \to (\lambda x, \lambda y, \lambda z)$.

$\mathbb{CP}^1$ has $K \simeq \mathcal{O}(-2)$, so there's a nonvanishing global section η of $K \otimes \mathcal{O}(2)$.

Then, the desired hol. 2-form on X is given by $\Omega = \pi^*(\eta) \wedge \frac{dx}{y}$.

Another way to view this: use homog. coords (u_1, u_2) for $\mathbb{CP}^1$; then we have five variables, ie $\mathbb{C}^5$, acted on by $\mathbb{C}^\times \times \mathbb{C}^\times$ with weights

x	y	z	u_1	u_2	
4	6	0	1	1	$[\Sigma = 12]$
1	1	1	0	0	$[\Sigma = 3]$

The quotient defines a "weighted projective space" $W = \mathbb{C}^5 / \mathbb{C}^\times \times \mathbb{C}^\times$. It has line bundles $\mathcal{O}(m,n)$ for $m,n \in \mathbb{Z}$. Canonical bundle $K = \mathcal{O}(-12,-3)$.
(the sums of the $\mathbb{C}^\times$-weights above) [Pf: use Euler seq. as for $\mathbb{CP}^n$]

And we defined X as the vanishing locus of a section of $\mathcal{O}(12,3)$.

$\Longrightarrow$ Using normal bundle seq. as before, get $c_1(X) = 0$ as desired.

So, we've built a K3 surface as a torus fibration, $\pi : X \to \mathbb{CP}^1$.
But the tori $\pi^{-1}(u)$ are complex, not sLag!

On the other hand, they are Lagrangian for Ω.
(In loc. coord on $\mathbb{CP}^1$, $\Omega = du \wedge \frac{dx}{y}$, and du pulls back to 0 on $\pi^{-1}(u)$)

Or, better said: we have 3 natural real 2-forms in the story,

$\omega_1 = \operatorname{Re} \Omega$

$\omega_2 = \operatorname{Im} \Omega$

$\omega_3 = \omega$ (Kähler form — for the Ricci-flat metric promised by Yau's Thm)

A special Lag. (w/ $\vartheta = 0$) has $i^*\omega_2 = i^*\omega_3 = 0$.

A complex Lag. ——— has $i^*\omega_1 = i^*\omega_2 = 0$.

To turn one into the other, want to somehow <u>permute</u> the roles of $\omega_{1,2,3}$.

How?