Riemannian Geometry: Exercise Set 1

Exercise 1

All the equations in this exercise can be summarized by the mnemonic that "if C is the change-of-basis matrix, then down indices transform by C, up indices transform by C^{-1}."

1. Let V be a real n-dimensional vector space. Let $\{e_i\}_{i=1}^n$ be a basis for V. Show that there is a basis $\{e^j\}_{j=1}^n$ for V^* with $e^j \cdot e_i = \delta_i^j$. We call this the "dual basis" to $\{e_i\}$.
2. Suppose $\{e_{i'}\}_{i'=1}^n$ is another basis for V, and let C be the change-of-basis matrix, i.e. the unique matrix such that
$$e_{i'} = C_{i'}^i e_i. \tag{0.1}$$
(In this equation there is an implicit $\sum_{i=1}^n$ on the right side: this is an example of the so-called "Einstein summation convention" which says that any index which appears once up and once down is to be summed over. We use this convention from now on.) Let $\{e^{j'}\}_{j'=1}^n$ be the dual basis to $\{e_{i'}\}_{i=1}^n$. Let C^{-1} be the inverse matrix to C, i.e.
$$C_{i'}^j (C^{-1})_i^{i'} = \delta_i^j, \qquad (C^{-1})_i^{i'} C_{j'}^i = \delta_{j'}^{i'}. \tag{0.2}$$
Then, show that
$$e^{j'} = (C^{-1})_j^{j'} e^j. \tag{0.3}$$
3. Let $T_k^l(V) = V^{\otimes k} \otimes (V^*)^{\otimes l}$. Suppose $w \in T_k^l(V)$. Let $w_{j_1 \cdots j_l}^{i_1 \cdots i_k} \in \mathbb{R}$ be the expansion coefficients of w with respect to the basis of $T_k^l(V)$ induced by $\{e_i\}_{i=1}^n$, i.e.
$$w = w_{j_1 \cdots j_l}^{i_1 \cdots i_k} (e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k})(e^{j_1} \otimes e^{j_2} \otimes \cdots e^{j_l}). \tag{0.4}$$
Let $w_{j_1' \cdots j_l'}^{i_1' \cdots i_k'} \in \mathbb{R}$ similarly be the expansion coefficients of w with respect to the basis $\{e'_{i'}\}$. Show that
$$w_{j_1' \cdots j_l'}^{i_1' \cdots i_k'} = w_{j_1 \cdots j_l}^{i_1 \cdots i_k} (C_{j_1'}^{j_1} C_{j_2'}^{j_2} \cdots C_{j_l'}^{j_l})((C^{-1})_{i_1}^{i_1'} (C^{-1})_{i_2}^{i_2'} \cdots (C^{-1})_{i_k}^{i_k'}). \tag{0.5}$$
(Suggestion: do this first in the cases $k = 1, l = 0$ and $k = 0, l = 1$.)

Exercise 2

Given an element $w \in T_k^l(V)$ with $k, l \geq 1$, the "trace of w in the last slots" is an element $\operatorname{Tr} w \in T_{k-1}^{l-1}(V)$, which can be defined as follows. First choose a basis for V. With respect to this basis w has components $w_{j_1 \cdots j_l}^{i_1 \cdots i_k}$. Now define $\operatorname{Tr} w$ to have components $w_{j_1 \cdots j_{l-1} i}^{i_1 \cdots i_{k-1} i}$. (Of course, we could similarly define the trace of w in the first slots, or in any pair of slots we like.)

1. Show that $\operatorname{Tr} w$ is well defined, independent of the basis we chose. (You could do this either by choosing another basis and checking it gives the same $\operatorname{Tr} w$, or by giving a direct basis-independent reformulation of the definition of $\operatorname{Tr} w$; it is enlightening to do it both ways.)

Exercise 3

1. Construct a canonical isomorphism

$$\mu : T^1_1(V) \to \mathrm{End}(V). \tag{0.6}$$

   (Of course, there exist many such isomorphisms, since both sides are vector spaces of dimension n^2. So the word *canonical* is important here. One way of understanding what it means is that μ should not depend on choosing a basis for V — or if you insist on choosing a basis, you have to show that μ is independent of the basis you chose.)
2. *Now* choose a basis $\{e_i\}_{i=1}^n$ for V. Relative to this basis, elements of $\mathrm{End}(V)$ are represented by $n \times n$ matrices. What is the matrix representing $\mu(e_i \otimes e^j)$?
3. Show that the "trace" defined in Exercise 2 goes over to the usual trace under the isomorphism μ.

Exercise 4

Let M be a smooth manifold, with an atlas $\{(U_\alpha, x_\alpha)\}$.

1. On each patch U_α we have a basis of sections of the tangent bundle TM, given by the coordinate vector fields $\{\partial/\partial x^i_\alpha\}_{i=1}^n$. We call this the *coordinate basis.* Show that the change-of-basis matrix relating two coordinate bases is

$$(C_{\alpha\alpha'})^{i'}_i = \frac{\partial x^{i'}_{\alpha'}}{\partial x^i_\alpha}. \tag{0.7}$$

   (Naturally, your solution will depend on your definition of TM.)
2. Suppose $M = S^2$. Describe TM by explicit patches and transition maps.
3. Now go back to arbitrary M with a given atlas. Describe the vector bundle $T^1_1 M$ by patches and transition maps. (Hint: Use the results of Exercise 1.)
4. Describe the vector bundle $T^2 M$ by patches and transition maps. (Hint: Use the results of Exercise 1.)
5. Describe the vector bundle $\wedge^2 T^* M$ of 2-forms on M by patches and transition maps. (Hint: Use the results of the previous part.)

Exercise 5

Let M be a smooth manifold.

1. Suppose X is a vector field on M and f a function on M. Show that in a coordinate basis

$$Xf = X^i \partial_i f. \tag{0.8}$$

2. Recall that the *differential* of a function f on M is a 1-form $\mathrm{d}f$, characterized by

$$(\mathrm{d}f)(X) = X(f) \tag{0.9}$$

for any vector field X. (Why is there a 1-form with this property?) Show that the components of $\mathrm{d}f$ in a coordinate basis are

$$(\mathrm{d}f)_i = \partial_i f. \tag{0.10}$$

3. Recall that the *Lie bracket* of two vector fields X, Y is a vector field $[X, Y]$, characterized by

$$[X, Y]f = X(Y(f)) - Y(X(f)). \tag{0.11}$$

(Why is there a vector field with this property?) Show that the components of $[X, Y]$ in a coordinate basis are

$$[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i. \tag{0.12}$$

4. We could try to define another vector field $[X, Y]_{wrong}$ by modifying this formula, say to

$$[X, Y]^i_{wrong} = X^j \partial_j Y^i - 2Y^j \partial_j X^i. \tag{0.13}$$

Show that this doesn't work: in other words, show that (for general enough X and Y) there is *no* vector field $[X, Y]_{wrong}$ with this coordinate expression.

Exercise 6

1. Let M be a smooth manifold. Show that M admits a Riemannian metric g. (Hint: use a partition of unity.)