Riemannian Geometry: Exercise Set 4

Exercise 1

Suppose E is a line bundle over M, i.e. a vector bundle of rank $r = 1$. Let ∇ be a connection in E. Note that $\operatorname{End} E$ is canonically trivial (a basis is provided by the identity map $1 \in \mathcal{E}(\operatorname{End} E)$), whatever E is.

1. Suppose that E is globally trivializable over M, and choose a global trivialization $s \in \mathcal{E}(E)$. Let $A \in \mathcal{E}(\operatorname{End} E \otimes T^*M) \simeq \mathcal{E}(T^*M)$ be the connection 1-form representing ∇ with respect to this trivialization. Show that
$$P_{\nabla,\gamma}(s(\gamma(0))) = e^{-\int_\gamma A} s(\gamma(1)). \tag{0.1}$$
2. Suppose that E is globally trivializable over M. Suppose γ is a closed path in M which is the boundary of some 2-chain C. Show that
$$P_{\nabla,\gamma} = e^{-\int_C F_\nabla}. \tag{0.2}$$
3. Repeat the previous part *without* the assumption that E is globally trivializable.
4. Suppose ∇ is flat. Show that $P_{\nabla,\gamma}$ depends only on the homology class of γ.
5. In class we proved that for E of arbitrary rank and ∇ flat, $P_{\nabla,\gamma}$ depends only on the *homotopy* class of γ. In the last part you showed the stronger statement that if E is a line bundle then $P_{\nabla,\gamma}$ depends only on the *homology* class of γ. Re-derive this result directly from the statement about homotopy, using the relation between π_1 and H_1.

Exercise 2

Suppose E is a vector bundle with a metric, i.e. we are given $g \in Sym^2(E^*)$. Let $\langle , \rangle$ be the induced bilinear pairing on $\mathcal{E}(E)$. Suppose ∇ is an *orthogonal* connection in E, i.e. $\mathrm{d}\langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle$ for any $s, s' \in \mathcal{E}(E)$.

1. Show that $\nabla g = 0$.
2. Show that the parallel transport operator $P_{\nabla,\gamma}$ is orthogonal, i.e. $\langle e, e' \rangle = \langle P_{\nabla,\gamma}(e), P_{\nabla,\gamma}(e') \rangle$. In particular, if γ begins and ends at the same point $x \in M$, then $P_{\nabla,\gamma}$ belongs to the orthogonal group $O(E_x)$.

Exercise 3

For any vector space V with nondegenerate bilinear pairing $\langle , \rangle$, define $o(V) \subset \operatorname{End} V$ to consist of those matrices B with
$$\langle Bv, v' \rangle = -\langle v, Bv' \rangle. \tag{0.3}$$
Equivalently, in an orthogonal basis for V, $o(V)$ consists exactly of the skew-symmetric matrices. (Indeed $o(V)$ is the Lie algebra of the orthogonal group $O(V)$.)

Now suppose given a bundle E over M. Let $o(E)$ be the subbundle of $\operatorname{End} E$ whose fiber at $x \in M$ is $o(E_x)$.

1. Suppose given two orthogonal connections ∇, ∇' in E. Let $A = \nabla' - \nabla \in \mathcal{E}(\operatorname{End} E \otimes T^*M)$. Show that that $A \in \mathcal{E}(o(E) \otimes T^*M)$. (So, in an orthogonal basis for E, A would be represented by an *antisymmetric* matrix of 1-forms.)
2. Suppose given an orthogonal connection ∇ in E. Show that $F_\nabla \in \mathcal{E}(o(E) \otimes \wedge^2 T^*M)$.