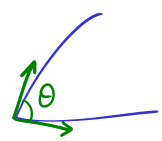
Intro

In diff. top. you have studied the notion of smooth manifold.
It's something that looks "locally like $\mathbb{R}^n$".
In ptic, locally, any two smooth manifolds look identical
So e.g. by "local measurements" using only the structure of smooth mfd, can't distinguish $\mathbb{R}^2$ from S^2.

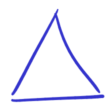
This doesn't capture many of the features of manifolds we see in the real world.

In particular: suppose you are an ant living on some general $M \subset \mathbb{R}^3$.
Then, you could distinguish M from $\mathbb{R}^2$. How?

Define
Length of a path: $\gamma : [0,T] \to M \subset \mathbb{R}^3$ $\quad L(\gamma) = \int_0^T dt \, \|\dot{\gamma}\|$ where $\|v\| = \sqrt{v \cdot v}$
with $\cdot$ the usual dot-prod in $\mathbb{R}^3$

Define angle between two paths:  $\quad \dot{\gamma}_1 \cdot \dot{\gamma}_2 = \|\dot{\gamma}_1\| \|\dot{\gamma}_2\| \cos\theta$

Define geodesics to be paths on M which locally minimize distance between two points.

Then, study geodesic triangles on M.  Let C be the sum of the interior angles.
You will find that $C \neq \pi$ in general.

For example: if $M = S^2$ of radius R, for a triangle of area A, find $C = \pi + \dfrac{A}{R^2}$

In general we may define $S(p) = \lim_{A \to 0} (C - \pi)/A$  ("scalar curvature").

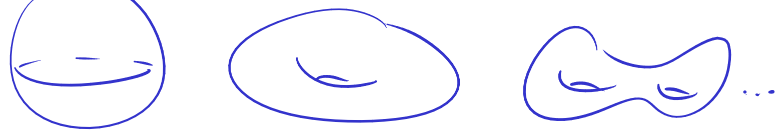
Then

$$S(p) = \begin{cases} 1/R^2 & \text{if } M = S^2 \text{ of radius } R \\ 0 & \text{if } M = \mathbb{R}^2 \end{cases}$$

So, this is a local invariant of $M \subset \mathbb{R}^3$. We defined it using the notion of length and angle inherited from $\mathbb{R}^3$.

Amazing fact [Gauss-Bonnet]:

$$\chi(M) = \frac{1}{4\pi}\int_M S \, dA$$



$\chi = 2$ $\chi = 0$ $\chi = -2$

so the ants living on M can determine its global topology just by making local measurements!

Our main aim in this course is to understand this "curvature" and its higher-dimensional analogues. For this we will need to study manifolds equipped with notions of dot-product of tangent vectors, $g(x): T_xM \otimes T_xM \to \mathbb{R}$ symmetric positive definite.

aka, Riemannian metrics.

[NB, this isn't the only possible notion: more generally one could have just a norm $F(x): T_xM \to \mathbb{R}$, "Finsler metric"]

In the example we just discussed, g was inherited: $TM \subset T\mathbb{R}^3$

$T\mathbb{R}^3$ has standard metric g_{can}, $\qquad g_M = g_{can}\big|_{TM \otimes TM}$

But in many cases M will not be a submanifold of anything — define g in some other way.

Curvature will turn out to be a 4-tensor $R \in T^3_1(M)$, subject to constraints so that it has $\frac{1}{12}n^2(n^2-1)$ independent components, e.g.

$n=1$: 0 $\quad n=3$: 6

$n=2$: 1 $\quad n=4$: 20 ...

The information in this tensor is contained equivalently in sectional curvature $K: Gr_2(TM) \to \mathbb{R}$ (roughly, curvature of "2-plane sections" of M).

Why study Riemannian metrics?

1) Many natural M come with natural g.

2) Studying g sometimes gives info about M: e.g.

a) Gauss-Bonnet-Chern: M cpt $\Rightarrow \chi(M) = \frac{1}{(2\pi)^n}\int_M \mathrm{Pf}(R)$

an n-form built algebraically from R

b) Cartan-Hadamard: if M is simply connected and admits a metric g with all sectional curvatures ≤ 0, then M is diffeo. to $\mathbb{R}^n$.

c) Hodge: g determines Laplacian operators $\Delta_k : \Omega^k(M,\mathbb{R}) \to \Omega^k(M,\mathbb{R})$,

and we have canonically $\ker \Delta_k \simeq H^k(M,\mathbb{R})$

d) Riem. metrics are the key tool in Perelman's pf of Poincaré conjecture:

M simply connected compact, $\dim M = 3 \implies M$ is homeomorphic to S^3.

⋮

3) Riem geometry (or a very slight generaliz^n, semi-Riem geometry, where we loosen the req^t of positive definiteness) occurs in nature: indeed spacetime is a semi-Riem manifold, and its curvature is responsible for gravity!

Hope to be able to say something about all of these topics.

Prereq: rudiments of differential topology

- smooth manifold
- vector bundle
- tangent, cotangent, tensor bundles
- Lie derivative

[See 2 of Lee has a very brief review.]
