Model spaces

Fundamental example: $S^n_R = \{x : \|x\|^2 = R^2\} \subset \mathbb{R}^{n+1}$

$O(n+1)$ acts on S^n_R by isometries, b/c it acts on $\mathbb{R}^{n+1}$ by isometries and preserves S^n_R (exercise).

In particular $\mathrm{Isom}(S^n_R)$ acts transitively on S^n_R.

Def (M,g) is *homogeneous* if $\mathrm{Isom}(M,g)$ acts transitively on M.

So, S^n_R is homogeneous.

Moreover:

A map $\varphi : M \to M$ has $d\varphi_p : T_pM \to T_{\varphi(p)}M$

If $p = \varphi(p)$, then $d\varphi_p \in \mathrm{End}(T_pM)$.

Def (M,g) is *isotropic at p* if $\mathrm{Stab}(p) \subset \mathrm{Isom}(M,g)$ acts transitively on unit sphere in T_pM.

For $M = S^n_R$, $\mathrm{Stab}(p) \subset O(n+1) \simeq O(T_pM)$ which does act transitively.

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right)$$

So, S^n_R is isotropic at every point.

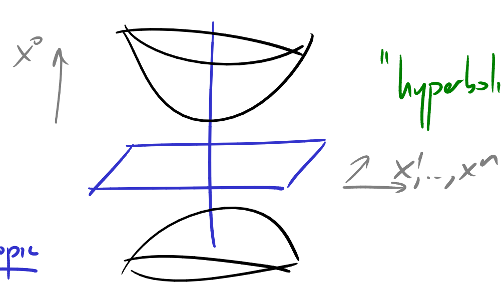
In summary: S^n_R is homogeneous isotropic.

similarly $\mathbb{R}^n$ is homogeneous isotropic (exercise).

Another fundamental example: let $\mathbb{R}^{n,1}$ mean $\mathbb{R}^{n+1}$ equipped with indefinite $\langle , \rangle$:

$$\langle x,y \rangle = (x^1y^1 + \cdots + x^ny^n) - x^0y^0$$

$H^n_R = \{x : \|x\|^2 = -R^2,\ x^0 > 0\} \subset \mathbb{R}^{n,1}$



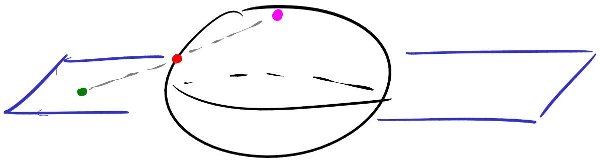
"hyperbolic n-space"

Induced metric on H^n_R is Riemannian (exercise).

$O_+(n,1)$ acts by isometries $\rightsquigarrow$ H^n_R is homogeneous isotropic

Writing the metric on S^n_R concretely: stereographic coordinates

$\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$



$$S^n \setminus \{(\vec{0},R)\} \xrightarrow{\sim} \mathbb{R}^n$$

$x \longmapsto$ the unique u s.t. $(u,0) \in \mathbb{R}^{n+1}$ is on the line connecting x and $(\vec{0},R)$

The inverse map is $x = \left(\dfrac{2R^2 u}{\|u\|^2+R^2},\ R\,\dfrac{\|u\|^2-R^2}{\|u\|^2+R^2}\right)$

Let's compute the metric on S^n in the coordinates (u^i):

then $dx^i = \dfrac{2R^2\,du^i}{\|u\|^2+R^2} - \dfrac{4R^2 u^i\, u\cdot du}{(\|u\|^2+R^2)^2}$ $\quad (1 \le i \le n)$, $\quad dx^{n+1} = R\,\dfrac{2u\cdot du}{\|u\|^2+R^2} + R\,\dfrac{\|u\|^2-R^2}{(\|u\|^2+R^2)^2}\,2u\cdot du$

$$= \frac{4R^3\|u\|^2\, u\cdot du}{(\|u\|^2+R^2)^2}$$

$$dx\cdot dx = \frac{4R^4}{(\|u\|^2+R^2)^2}\,du\cdot du + \frac{16R^4\,\|u\|^2\,(u\cdot du)^2}{(\|u\|^2+R^2)^4} - \frac{16R^4(u\cdot du)^2}{(\|u\|^2+R^2)^3} + \frac{16R^6\,\|u\|^4\,(u\cdot du)^2}{(\|u\|^2+R^2)^4}$$

$$dx\cdot dx = \frac{4R^4}{(\|u\|^2+R^2)^2}\,du\cdot du$$

[i.e. in the coord basis given by the (u^i),
$$(g_{S^n})_{ij} = \frac{4R^4}{(\|u\|^2+R^2)^2}\,\delta_{ij}$$]

In other words:

Def (M,g) is conformal to $(M,\tilde{g})$ if $g = f\tilde{g}$ for some $f \in C^\infty(M)$.

[Rk: this $\iff$ g and $\tilde{g}$ define the same angles between vectors.]

Def (M,g) is locally conformally flat if each pt. has a nbhd which is conformal to a ball in $\mathbb{R}^n$

We showed S^n_R is locally conformally flat.

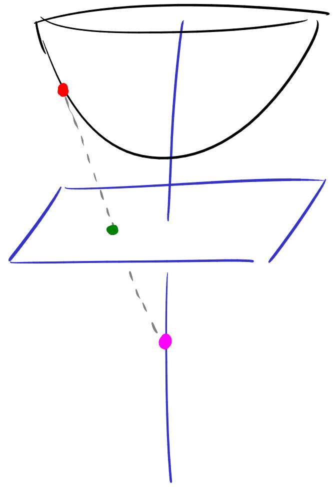
Def A map $\varphi: (M,g) \longrightarrow (\tilde{M},\tilde{g})$ is conformal if $\varphi^*\tilde{g}$ and g are conformal.

Rk: As we go toward ∞ in the u-coordinate, the rescaling factor $\dfrac{4R^4}{(\|u\|^2+R^2)^2} \longrightarrow 0$.

In ptic, length of a path $\gamma(t) = (t,0,\ldots,0)$ in u-coord: $\displaystyle\int_0^L \|\dot\gamma\| = \int_0^L \frac{2R^2}{\|u\|^2+R^2}\,dt = 2R\tan^{-1}\!\left(\frac{L}{R}\right)$ $\to \pi R$ as $L \to \infty$

Similarly for $\mathbb{H}^n_R$:

$\mathbb{B}^n_R = \{\|x\| < R\} \subset \mathbb{R}^n$

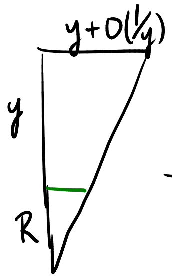


Stereographic projection:

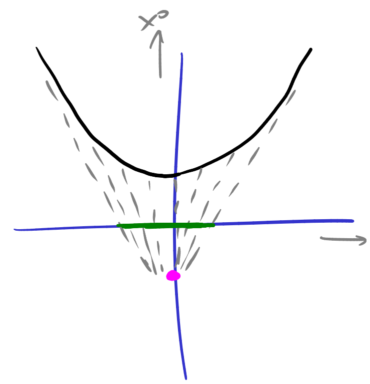
$\mathbb{H}^n_R \xrightarrow{\sim} \mathbb{B}^n_R$

$x \mapsto u \in \mathbb{R}^n$ s.t. $(u,0)$ is on the line with x and $(0,-R)$

(To see that the image is $\mathbb{B}^n_R$, fix $x^2 = \cdots = x^n = 0$, then



$\frac{Ry}{y+R} + O(\frac{1}{y}) \longrightarrow R$ as $y \to \infty$



)

<u>Global</u> coord. in this case, so it actually <u>identifies</u> $\mathbb{H}^n_R$ with $\mathbb{B}^n_R$;

Then compute directly as for S^n, $dx \cdot dx = \frac{4R^4}{(R^2 - \|u\|^2)^2} du \cdot du$

ie $u(x)$ gives an <u>isometry</u> between $\mathbb{H}^n_R$ and $\left(\mathbb{B}^n_R, \frac{4R^4}{(R^2-\|u\|^2)^2} g_{can}\right)$

call the latter the "Poincare ball model" of $\mathbb{H}^n_R$